1023
69
Twyle

TRANSACTIONS

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

WILLIAM FOGG OSGOOD

EDWARD BURR VAN VLECK HENRY SEELY WHITE

WITH THE COÖPERATION OF

GILBERT A. BLISS CHARLES L. BOUTON LEONARD E. DICKSON

JOHN I. HUTCHINSON EDWARD KASNER FOREST R. MOULTON

ERNST J. WILCZYNSKI EDWIN B. WILSON

VOLUME 11

1910

PUBLISHED BY THE SOCIETY

LANCASTER, PA., AND NEW YORK

1910

N

510.6
AMS
v. 11.

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA

TABLE OF CONTENTS

VOLUME 11 1910

PAGES

BECK, H., of Charlottenburg, Prussia. Ein Seitenstück zur Mobius'schen Geometrie der Kreisverwandtschaften 414

BIRKHOFF, G. D., of Princeton, N. J. A simplified treatment of the regular singular point 199

BLICHFELDT, H. F., of Stanford University, Cal. Theorems on simple groups 1

BLISS, GILBERT AMES, of Chicago, Ill. (with MAX MASON). Fields of extremals in space 325

CRAIG, CLYDE FIRMAN, of Ithaca, N. Y. On a class of hyperfuchsian functions 37

EISENHART, LUTHER PFAHLER, of Princeton, N. J. Congruences of the elliptic type 351

Surfaces with isothermal representation of their lines of curvature and their transformations (second memoir) 475

EVANS, GRIFFITH C., of Cambridge, Mass. Volterra's integral equation of the second kind, with discontinuous kernel 393

HOSKINS, L. M., of Stanford University, Cal. The strain of a gravitating, compressible elastic sphere 203

The strain of a non-gravitating sphere of variable density . 494

INGOLD, LOUIS, of Columbia, Mo. Vector interpretation of symbolic differential parameters 449

KASNER, EDWARD, of New York, N. Y. The theorem of Thomson and Tait and natural families of trajectories 121

LYTLE, ERNEST B., of Urbana, Ill. Proper multiple integrals over iterable fields 25

MACMILLAN, WILLIAM DUNCAN, of Chicago, Ill. Periodic orbits about an oblate spheroid 55

MASON, MAX, of Madison, Wis. (with G. A. BLISS). Fields of extremals in space 325

MILLER, G. A., of Urbana, Ill. Groups generated by two operators (s_1, s_2) satisfying the equation $s_1 s_2^2 = s_2 s_1^2$ 341

NEIKIRK, LEWIS IRVING, of Urbana, Ill. Groups of rational transformations in a general field 294

OWENS, F. W., of Ithaca, N. Y. The introduction of ideal elements and a new definition of projective n-space 141

198114

PAGES

RANUM, ARTHUR, of Ithaca, N. Y. The groups of classes of congruent quadratic integers with respect to a composite ideal modulus . . . 172

SMITH, PERCEY F., of New Haven, Conn. On osculating element-bands associated with loci of surface-elements 301

SNYDER, VIRGIL, of Ithaca, N. Y. Infinite discontinuous groups of birational transformations which leave certain surfaces invariant . 15

———— Conjugate line-congruences contained in a bundle of quadric surfaces 371

STUDY, EDUARD, of Bonn, Germany. Die natürlichen Gleichungen der analytischen Curven im Euclidischen Raume 249

WAHLIN, G. E., of Urbana, Ill. On the base of a relative number-field, with an application to the composition of fields 487

WESTLUND, JACOB, of Lafayette, Ind. On the fundamental number of the algebraic number-field $(k(\sqrt[p]{m})$ 388

YOUNG, JOHN WESLEY, of Lawrence, Kans. Two-dimensional chains and the associated collineations 280

THEOREMS ON SIMPLE GROUPS*

BY

H. F. BLICHFELDT

Introduction and Terminology.

§1. It is our purpose in this paper to state and prove a somewhat more precise theorem on simple groups than that given by FROBENIUS in his article *Ueber auflösbare Gruppen. V.*† This theorem of FROBENIUS we shall state in the following form: "Let H be a group whose order is divisible by p^λ, p being a prime. If H is simple, then there is in H a substitution T_1 whose order is prime to p, which is commutative with some subgroup Q of H of order p^μ and not commutative with every subgroup of Q. In the contrary case, H has an invariant subgroup of index p^λ, containing all the substitutions of H whose orders are prime to p."

The theorem which we shall here establish (divided for convenience into four parts, I, II, III, IV) defines the group Q more explicitly, and the nature of the relation of T_1 to Q in the most important case, namely, when Q is a Sylow subgroup of H. Several applications are added in the form of corollaries.

The phraseology of ordinary group theory will be employed. The following abbreviations are used:

$A = B$ means "the groups (substitutions) A and B are identical";

$A < B$ means "the group (substitution) A is contained in the group B, though it is not identical with B."

The letters H, P, P_1 and T have the following significance throughout:

H represents a group of order $p^\lambda n$, p being a prime number not dividing n;

P represents any given subgroup of H of order p^λ;

P_1 represents any given subgroup of P of order $p^{\lambda-1}$; and

T is the general symbol for a substitution of H whose order is prime to p.

Though it is contrary to general usage, we shall proceed from right to left in a succession of substitutions indicated. A function subjected to a substitution or a succession of such shall be written to the right of the substitutions indi-

* The present paper is an extension of two papers read before the San Francisco section, viz., *A theorem concerning the Sylow subgroups of simple groups*, September 29, 1906; and *A theorem on simple groups*, September 26, 1908.

†Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 1901, p. 1324.

cated, enclosed in parentheses. Thus, if A and B represent substitutions, I a function, the symbol $AB(I)$ represents the function obtained by first subjecting I to the substitution B, and then subjecting the result to the substitution A.

Theorems and corollaries.

§ 2. **Theorem I.** *Among the substitutions contained in P, but not in P_1, let S be any one of those of lowest order. Then, if H is simple, there are in H a group Q (either $= P$ or $< P$, containing S) and a substitution T, say T_1, of the following nature:*

1°) *If $Q = P$, this substitution T_1 is commutative with P but not with S; more generally, the commutator $S^{-1}T_1^{-1}ST_1$ is not in P_1.*

2°) *If $Q < P$, then T_1 is commutative with Q, but not with every subgroup of Q of index p.*

Theorem II. *If $Q < P$, there is a group $Q_1 \leqq Q$ containing S, and there is a certain series of groups*

$$Q_1,\ Q_2,\ Q_3,\ \cdots,\ Q_{\lambda-\nu+1} = P$$

of orders $p^\nu, p^{\nu+1}, p^{\nu+2}, \cdots, p^\lambda$, respectively, each a subgroup of the one to the right of it, possessing the following property. Let R_i be that group whose substitutions are common to all the subgroups of Q_i of index p. Then, if A_i be any substitution of Q_i but not of Q_{i-1}, $i > 1$, the substitution

$$(SA_i)^p A_i^{-p},$$

though a substitution of Q_{i-1}, does not belong to R_{i-1}.

Corollary 1. *Every substitution of P commutative with S must belong to Q.* More generally, if $p > 2$, every substitution A of P, such that the substitutions A and SAS^{-1} are commutative, must belong to Q. Hence, *if $p > 2$, every abelian subgroup of P which is transformed into itself by S, must belong to Q.*

We prove this corollary from Theorem II by making use of the facts that R_{i-1} is invariant in Q_i; that every commutator of Q_{i-1} belongs to R_{i-1}; that the pth powers of the substitutions of Q_{i-1} belong to R_{i-1}.

Theorem III.

α) If $Q_1 < P$, then the order p^λ of P is $\geqq p^{p+1}$, or is $\geqq p^{p^2}$, according as the order of Q_1 is $= p^{\lambda-1}$, or is $< p^{\lambda-1}$.

β) If the order of P is $< p^{2p-1}$, then there is in P some subgroup P_1 and substitution S for which the corresponding group $Q = P$.

Corollary 2. *The group Q is certainly $= P$ in the cases where P*

1°) *is abelian* (Theorem II),

2°) *or contains no substitutions of order p^2* (Theorem II),

3°) *or is of order p^p at most* (Theorem III).

In any of these cases H will therefore contain a subgroup M which contains P invariantively. Any one of the substitutions S of P (defined in Theorem I) is non-commutative with some substitution T of M.

In particular, if P fulfills both 1°) and 2°), then no one of its substitutions (except identity) is invariant in M.

Corollary 3. *Let p be the lowest prime which divides the order of A, and let it be given that the corresponding subgroup P is abelian.* Let a set of independent generating substitutions of P be constructed; then, if one of these generators be of order p^a, there must be at least three generators of order p^a if $p > 2$; at least two if $p = 2$.*

Hence, if $p > 2$, and if the order of P be $\leqq p^8$, then P must be of type $(1, 1, \cdots, 1)$ in every case, with the exception of the possibility $(2, 2, 2)$. Again, if $p = 2$, and the order of P be $\leqq p^6$, the only permissible types are the following: $(1, 1)$, $(1, 1, 1)$, $(1, 1, 1, 1)$, $(1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 1)$, $(2, 2)$, $(2, 2, 1, 1)$, $(2, 2, 2)$, $(3, 3)$.

To prove this corollary. let A_1, A_2, $\cdots$ be the generators of order p^a; B_1, B_2, $\cdots$ the generators whose orders, p^{b_1}, p^{b_2}, $\cdots$ are $> p^a$. The group K, generated by the substitutions $B_1^{p^{b_1-1}}$, $B_2^{p^{b_2-1}}$, $\cdots$, $A_1^{p^{a-1}}$, $A_2^{p^{a-1}}$, $\cdots$, is a characteristic subgroup of P.† Moreover, the group K', generated by $B_1^{p^{b_1-1}}$, $B_2^{p^{b_2-1}}$, $\cdots$ (leaving out of K the generators $A_1^{p^{a-1}}$, $\cdots$), is a characteristic subgroup of P. Let the factor-group K/K' be denoted by L, and its generating substitutions by $a_1, a_2, \cdots$, corresponding to $A_1, A_2 \cdots$.

In applying Theorem I, 1°), let P_1 be that group generated by those generators of P whose orders are higher or lower than p^a, and by $A_1^p, A_2, \cdots$. Then we may put $S = A_1$, and we must have

$$T_1^{-1} A_1 T_1 = A_1^x C, \qquad x \not\equiv 1 \pmod{p},$$

where C belongs to P_1.

Accordingly,

$$T_1^{-1} A_1^{p^{a-1}} T_1 = (A_1^{p^{a-1}})^x C^{p^{a-1}}.$$

Since T_1 is commutative with L, we have correspondingly

$$T_1^{-1} a_1 T_1 = a_1^r c,$$

c belonging to that subgroup of L generated by $a_2, \cdots$. Hence, if there were only two generators, a_1, a_2, in L, then the order of T_1 would be a factor of $p^2 - 1$. But this number is divisible by no prime $> p$ except when $p = 2$.

* The more general case where P is not abelian, if it be only such that, for every P_1 and S, $Q = P$ (cf. Cor. 2), is embodied in this corollary when $B_1, \cdots, A_1, \cdots$, instead of being the generators of P, are the generators of the factor-group P/W, W being the commutator subgroup of P.

† FROBENIUS, *Ueber auflösbare Gruppen*. II, Sitzungsberichte der Akademie der Wissenschaften zu Berlin (1895), pp. 1028–1029. BURNSIDE, *Theory of Groups*, pp. 233–235.

At the end of his paper on soluble groups, referred to in the introduction,* FROBENIUS proves that every group of order $p^\lambda qr$ is soluble, if $\lambda \leqq 4$; p, q, r being odd primes. Using his arguments in conjunction with Theorems I–III we find the

Corollary 4. *Every group of order $p^\lambda qr$ is soluble, if $\lambda \leqq p^2 - 1$; p, q, r being different odd primes.*

Theorem IV. *If the conditions as specified for a simple group H in Theorems* I, II *and* III *are not all fulfilled with reference to S and P, then H has an invariant subgroup H_1 of index p, which does not contain S, but contains every substitution T of H.*

Proof of the theorems.

§ 3. Let P' be the largest subgroup of H whose substitutions are commutative with P, and which, besides, transform S into itself or into $S\times$ (a substitution of P_1). This group P' is of order $p^\lambda n'$, n' being a factor of n. We shall prove that

A) The group P' contains an invariant subgroup P_1' of order $p^{\lambda-1}$, to which S does not belong; further, no substitution of P, not belonging to P_1', is of lower order than S.

We write P' as a regular group in $p^\lambda n'$ letters

$$x_1, x_2, \cdots, x_{p^\lambda n'}.$$

The substitutions of P_1 will transform x_1 into $p^{\lambda-1}$ different letters, say

$$x_1, x_2, \cdots, x_{p^{\lambda-1}}.$$

The functions

$$(1) \qquad I_1 = x_1 + x_2 + \cdots + x_{p^{\lambda-1}},\ S(I_1),\ S^2(I_1),\ \cdots,\ S^{p-1}(I_1)$$

are all absolute invariants for the group P_1. Let θ be a root different from 1, of the equation

$$\theta^p - 1 = 0,$$

and let us consider the function

$$I_2 = I_1 + \theta^{-1}S(I_1) + \theta^{-2}S^2(I_1) + \cdots + \theta^{-p+1}S^{p-1}(I_1).$$

It is not identically zero, no two of the functions (1) containing the same letter x_i. It is an absolute invariant for the group P_1, and we have

$$S(I_2) = \theta I_2.$$

We shall introduce the following abbreviations:

ζ means a power of θ; and

* Loc. cit., p. 1329.

ϕ means a power of θ, different from 1.

Then if A is any substitution of P,

$$A(I_2) = \zeta I_2, \qquad S(I_2) = \phi I_2.$$

§ 4. When the group P' is written symbolically in the form

$$P' = P + B_2P + B_3P + \cdots + B_{n'}P,$$

the function

$$I_3 = I_2 \cdot B_2(I_2) \cdot B_3(I_2) \cdot \cdots \cdot B_{n'}(I_2) \tag{2}$$

is seen to be a relative invariant for P'. We find that

$$S(I_3) = \theta^{n'} I_3 = \phi I_3.$$

It follows that all the substitutions of P' for which I_3 is an absolute invariant form an invariant subgroup P'' of order $p^{\lambda-1}n'$, not containing S. This subgroup has a subgroup P_1' of order $p^{\lambda-1}$, also invariant in P'. The first part of (A), § 3, has thus been proved.

To prove the second part, we remark first that, as P is invariant in P', all the factors of I_3 are relative invariants for P. Hence, if A be a substitution of P and not of P_1', the condition

$$A(I_3) = \phi I_3$$

necessitates that, for some factor $B_i(I_2)$ of the right-hand member of (2), the relation shall subsist:

$$AB_i(I_2) = \phi B_i(I_2).$$

This gives

$$B_i^{-1}AB_i(I_2) = \phi I_2.$$

It follows that $B_i^{-1}AB_i$, though a substitution of P, does not belong to P_1. Hence its order is not lower than that of S (cf. Theorem I). Hence the order of A is not lower than that of S.

§ 5. We shall now write H as a regular group in $h = p^\lambda n$ letters, $x_1, x_2, \cdots, x_h$. The substitutions of P' will transform x_1 into $p^{\lambda-1}n'$ letters, say $x_1, x_2, \cdots, x_{p^{\lambda-1}n'}$. Let $I = x_1 + x_2 + \cdots + x_{p^{\lambda-1}n'}$, and let us consider the function

$$J = I + \theta^{-1}S(I) + \theta^{-2}S^2(I) + \cdots + \theta^{-p+1}S^{p-1}(I),$$

which is not identically zero.

B) The group P' contains all the substitutions of H for which J is a relative invariant.

Let W transform x_a into x_b, both occurring in J. Since the letters involved in J form a transitive set for the group P', we can find a substitution in the latter, say V, which also transforms x_a into x_b. The substitution VW^{-1} will

therefore leave x_b unchanged. Since H is written in regular form, it follows that VW^{-1} is identity. *If J be a relative invariant for W, and the order of W be a power of p, then W belongs to P.*

§ 6. The substitutions of H will transform J into n/n' expressions that do not differ by constant factors merely. The product π of these factors will be a relative or absolute invariant for H. Evidently, $A(\pi) = \zeta\pi$, A being any substitution of H. If

$$S(\pi) = \phi\pi,$$

then H contains an invariant subgroup H_1 of index p, consisting of all the substitutions C for which

$$C(\pi) = \pi.$$

To this subgroup would belong every substitution T of H. Accordingly, *unless the group H_1 of Theorem IV exists, we must have*

$$S(\pi) = \pi. \tag{3}$$

§ 7. To examine the nature of the factors of π, let us write H symbolically in the form

$$H = P' + PV_1 + PV_2 + \cdots + PV_{n-n'}. \tag{4}$$

The group P', operating upon J, will furnish just one of the factors of π, namely J. The p^λ substitutions represented by the symbol PV_i will furnish, say, p^{k_i} factors of π, whose product we shall indicate by π_i. Now, for $j \neq i$, one of the factors of π_j may be equal, apart from a constant multiplier, to one of the factors of π_i. In such a case we find $\pi_j = \pi_i \times$ (a constant). We shall then agree to say that $\pi_j = 1$, if $j > i$. With this understanding we may write

$$\pi = J\pi_1\pi_2 \cdots \pi_{n-n'}.$$

Since $S(J) = \theta J$, $S(\pi_i) = \zeta\pi_i$, it follows that, if (3) is to hold, there is at least one index ρ such that

$$S(\pi_\rho) = \phi\pi_\rho. \tag{5}$$

§ 8. First, let π_ρ consist of a single factor. Then the alternative case 1°) of Theorem I is proved in the following manner. Let V be one of the substitutions PV_ρ of (4). Then we may suppose that

$$\pi_\rho = V(J).$$

The function π_ρ is here a relative invariant for P. Hence, J is a relative invariant for $V^{-1}PV$. Hence, by (B), § 5,

$$V^{-1}PV = P.$$

Now, since V does not belong to P', the group generated by V and P', though containing P invariantively, does not fulfill the other conditions imposed upon P' in § 3. Case 1°) of Theorem I thus follows.

§ 9. Next, let π_ρ consist of more than one factor,

$$\pi_\rho = y_1 y_2 \cdots y_{p^k}. \tag{6}$$

C) Then we shall prove that *V, one of the substitutions of the symbol PV_ρ, transforms a certain subgroup Q_1 of P, which contains S, into itself or into another subgroup of P. Moreover, V does not transform that subgroup of Q_1 which is also a subgroup of P_1' into itself or into another subgroup of P_1'.*

Substituting (6) in (5), it might happen that, for some index j, we have $S(y_j) \neq \zeta y_j$. Then we may suppose that

$$S(y_j) = y_{j+1},\ S(y_{j+1}) = y_{j+2},\ \cdots,\ S(y_{j-1+p^l}) = \theta' y_j \qquad (\theta'^{p^l} = 1).$$

Thus, if $y_j = W(J)$, we get

$$W^{-1} S^{p^l} W(J) = \theta' J. \tag{7}$$

Hence (§ 5, *B*) $W^{-1} S^{p^l} W < P$.

This substitution, being of lower order than S, must belong to P_1' (§ 3, *A*). The function J is, however, an absolute invariant for P'' and therefore for P_1' (cf. §§ 4, 5). Accordingly $\theta' = 1$.

§ 10. The p^k factors of π_ρ (6), permuted transitively by P, fall into p sets of p^{k-1} factors each, forming imprimitive sets. Let the p products of the factors of these sets be indicated by

$$a_1,\ a_2,\ \cdots,\ a_p; \qquad a_1 a_2 \cdots a_p = \pi_\rho.$$

Substituting in (5), either we have

$$S(a_i) = \zeta a_i \qquad (i = 1, 2, \cdots, p); \tag{8}$$

or we may put

$$S(a_1) = a_2,\ S(a_2) = a_3,\ \cdots,\ S(a_p) = \phi a_1.$$

The latter relations would, however, lead to an equation like (7), $\theta' \neq 1$, contrary to what was proved in § 9. Accordingly, (8) is true, and $\zeta = \phi$ for some index i, say $i = 1$:

$$S(a_1) = \phi a_1. \tag{9}$$

That subgroup of P for which $a_1, a_2, \cdots, a_p$ are relative invariants shall be designated by $Q_{\lambda-\nu}$, where $\nu = \lambda - k$. Its order is $p^{\lambda-1}$.

The p^{k-1} factors of a_1 fall into p sets of imprimitivity for the group $Q_{\lambda-\nu}$. The respective products shall be indicated by

$$b_1,\ b_2,\ \cdots,\ b_p; \qquad b_1 b_2 \cdots b_p = a_1.$$

We may assume that $S(b_1) = \phi b_1$. The factors $b_1, b_2, \cdots, b_p$ are relative invariants for a group $Q_{\lambda-\nu-1} < Q_{\lambda-\nu}$, of order $p^{\lambda-2}$. Proceeding thus, we finally arrive at groups Q_3, Q_2, Q_1, of orders $p^{\nu+2}$, $p^{\nu+1}$, p^{ν} respectively, having sets of relative invariants respectively designated by

$$w_1, w_2, \cdots, w_p; \qquad v_1, v_2, \cdots, v_p; \qquad y_1, y_2, \cdots, y_p.$$

These invariants satisfy the relations

$$w_1 = v_1 v_2 \cdots v_p; \qquad v_1 = y_1 y_2 \cdots y_p.$$

It will be noticed that S belongs to all of these groups, and we have, corresponding to (9),

$$S(w_1) = \phi w_1, \quad S(v_1) = \phi v_1, \quad S(y_1) = \phi y_1.$$

§ 11. Let $y_1 = V(J)$, V being one of the p^{λ} substitutions PV_{ρ}[(4), § 7]. Then J is a relative invariant for the group $V^{-1}Q_1V$, which must therefore be a subgroup of P(§ 5). The first statement in (C), § 9, is thus true.

Let A be a substitution of Q_2, but not of Q_1. Then we may suppose that

$$y_i = A^{-i+1}(y_1) \qquad (i = 2, \cdots, p);$$

and we get

$$\phi = \frac{S(v_1)}{v_1} = \frac{S(y_1)}{y_1} \cdot \frac{S(y_2)}{y_2} \cdots = \frac{S \cdot ASA^{-1} \cdot A^2SA^{-2} \cdot \cdots \cdot A^{p-1}SA^{-p+1}(y_1)}{y_1}$$

$$= \frac{(SA)^p A^{-p}(y_1)}{y_1} = \frac{V^{-1}(SA)^p A^{-p} V(J)}{J}. \tag{10}$$

Now $(SA)^p A^{-p} < Q_1$, and is also contained in every subgroup of P of order $p^{\lambda-1}$. It is therefore contained in P_1'. For this group J is an absolute invariant. Therefore, if the second part of (C), § 9, were not true, then (10) could not be true.

§ 12. We shall now prove case 2°) of Theorem I. Let $Q^{(0)}$ be any subgroup of P such that $Q^{(0)} \geqq Q_1$, and let

$$Q^{(0)}, Q^{(1)}, Q^{(2)}, \cdots, Q^{(n)} = P \tag{11}$$

be that series of groups in which each member is the largest subgroup of P containing the one immediately to the left of it invariantively. If H contains a substitution T which is commutative with one of these groups, say $Q^{(m)}$, $m > 0$, without being commutative with $Q^{(m-1)}$, then 2°) of Theorem I is true; the group Q in the theorem representing a certain group $> Q^{(m-1)}$ and $\leqq Q^{(m)}$. We shall therefore assume that every substitution P which is commutative with $Q^{(m)}$ of (11), or of any similar series, is commutative also with Q^{m-1}, $m > 0$. Under

this assumption the following theorem by Frobenius holds: If K represents the greatest subgroup of H in which $Q^{(k)}$ is invariant ($n > k \geqq 0$), then $Q^{(k+1)}$ is a Sylow subgroup of K.*

D) By means of this theorem we can prove that, *if V be the substitution considered in* §§ 9, 11, *we may write*

$$V = W_1 W_2 W_3 \cdots W_r,$$

where W_1, W_2, $\cdots$, W_r are substitutions of H, respectively commutative with the following members of a certain series of groups:

$$L_1 = Q_1,\ L_2,\ L_3,\ \cdots,\ L_r = P, \tag{12}$$

each a subgroup of those that follow.

§ 13. To prove (*D*), let $VPV^{-1} = M$. Then M and P have in common the group Q_1, but no greater group. Let Q_1' be the greatest subgroup of P containing Q_1 invariantively, and Q_1'' the corresponding subgroup of M. The groups Q_1' and Q_1'' will generate a group N in which Q_1' is a Sylow subgroup by Frobenius's theorem, § 12. There is therefore in N a substitution W_1 such that $Q_{1,1} = W_1^{-1} Q_1'' W_1 \leqq Q_1'$, $Q_{1,1} > Q_1$. We have also $W_1^{-1} Q_1 W_1 = Q_1$. Let now $(W_1^{-1} V) P (W_1^{-1} V)^{-1} = M_1$. Then P and M_1 have in common a subgroup, say L_2, $\geqq Q_{1,1}$, and therefore $L_2 > Q_1$. If $L_2 = P$, then (*D*) is proved, the series (12) consisting of the two terms Q_1, P; in this case we find

$$V = W_1 W_2,$$

where W_2 is a certain substitution of H commutative with P.

If $L_2 < P$, let L_2' be the largest subgroup of P containing L_2 invariantively, L_2'' the corresponding subgroup of M_1. The groups L_2' and L_2'' generate a group in which L_2' is a Sylow subgroup, etc. Proceeding as above, the proof of (*D*) may be completed without difficulty.

§ 14. To complete the proof of case 2°) of Theorem I, let us consider the substitutions W_1, W_2, $\cdots$, W_r, respectively commutative with the groups of (12). If they were at the same time commutative with the corresponding subgroups of P_1', i. e., if W_i, which is commutative with L_i, is also commutative with that subgroup (L_i^0) of L_i which is a subgroup of P_1', then (*C*), § 9, would not be true. For, the group

$$V^{-1} L_1^0 V = W_r^{-1} \{ \cdots [W_2^{-1} (W_1^{-1} L_1^0 W_1) W_2] \cdots \} W_r$$

* Paper referred to in § 1, p. 1325, Lemma I. The above theorem is apparently more general than that given by FROBENIUS, account being taken of the underlying assumptions as stated in both places. The method of proof by FROBENIUS is valid for the case considered here.

would then be a subgroup of P_1'. Accordingly, for some subscript i, there is a substitution, W_i, commutative with L_i, but not with L_i^0.

Now if G be the largest subgroup of H in which L_i is invariant, then L_i' is a Sylow subgroup of G (§§ 12, 13). Evidently, L_i^0 is invariant in L_i'. Since L_i^0 is not invariant in G (W_i belongs to G), it follows that there is a substitution T in G which is not commutative with L_i^0. The group L_i takes the place of Q in Theorem I, 2°).

§ 15. To prove Theorem II, let

$$Q_1,\ Q_2,\ \cdots,\ Q_{\lambda-\nu},\ P$$

be the series of groups arrived at in § 10. Let $A_{\lambda-\nu+1}$ be a substitution belonging to P, but not to $Q_{\lambda-\nu}$. Then we may suppose that

$$a_i = A_{\lambda-\nu+1}^{-i+1}(a_1) \qquad (i=2, 3, \cdots, p).$$

We obtain

$$\phi = \frac{S(\pi_\rho)}{\pi_\rho} = \frac{1}{a_1}\left[S\cdot A_{\lambda-\nu+1} S A_{\lambda-\nu+1}^{-1}\cdot A_{\lambda-\nu+1}^{2} S A_{\lambda-\nu+1}^{-2}\cdot\cdots\cdot(a_1)\right]$$

$$= \frac{1}{a_1}\left(S A_{\lambda-\nu+1}\right)^p A_{\lambda-\nu+1}^{-p}(a_1).$$

The function a_1, being a relative invariant for $Q_{\lambda-\nu}$, is plainly an absolute invariant for the group $R_{\lambda-\nu}$, whose substitutions are common to all the subgroups of $Q_{\lambda-\nu}$ of order $p^{\lambda-2}$. It follows that $\left(S A_{\lambda-\nu+1}\right)^p A_{\lambda-\nu+1}^{-p}$ does not belong to $R_{\lambda-\nu}$.

Starting now with a substitution $A_{\lambda-\nu}$ belonging to $Q_{\lambda-\nu}$ but not to $Q_{\lambda-\nu+1}$, we may suppose

$$b_i = A_{\lambda-\nu}^{-i+1}(b_1) \qquad (i=2, 3, \cdots, p).$$

Substituting in $\phi = S(a_1)/a_1$, we find that $\left(S A_{\lambda-\nu}\right)^p A_{\lambda-\nu}^{-p}$ does not belong to $R_{\lambda-\nu-1}$, etc.

The group Q_1 is $\leqq Q$ of Theorem I, 2°), by §§ 12–14.

§ 16. We proceed to prove Theorem III. Considering the groups $Q_1, Q_2, \cdots, P$ of § 10, let A designate a substitution belonging to Q_2, but not to Q_1. We will assume that A permutes the letters $y_1, y_2, \cdots, y_p$ in the order

$$A: \qquad (y_1 y_2 \cdots y_p),$$

and that

$$S(y_i) = \theta^{\alpha_i}(y_i) \qquad (i=1, 2, \cdots, p).$$

Then, since $S(v_1) = \phi v_1$,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p \not\equiv 0 \pmod{p}.$$

We shall say that *the constants* $\theta^{\alpha_1}, \theta^{\alpha_2}, \ldots, \theta^{\alpha_p}$ *are the multipliers of* S.

The substitution

$$S' = S^{x_1} \cdot A^{-1} S^{x_2} A \cdot A^{-2} S^{x_3} A^2 \cdot \cdots \cdot A^{-p+1} S^{x_p} A^{p-1},$$

belonging to Q_1, will have the multipliers θ, 1, 1, $\cdots$, 1, provided the congruences

$$\begin{aligned} &\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p \equiv 1, \\ &\alpha_2 x_1 + \alpha_3 x_2 + \cdots + \alpha_1 x_p \equiv 0, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\alpha_p x_1 + \alpha_1 x_2 + \cdots + \alpha_{p-1} x_p \equiv 0 \pmod{p} \end{aligned}$$

are satisfied. The necessary condition is that the determinant Δ of the coefficients does not vanish (mod p). We find

$$\begin{aligned} \Delta &= (\alpha_1 + \theta\alpha_2 + \theta^2\alpha_3 + \cdots)(\alpha_1 + \theta^2\alpha_2 + \theta^4\alpha_3 + \cdots)\cdots \\ &\equiv \Delta^p \equiv (\alpha_1^p + \alpha_2^p + \alpha_3^p + \cdots)^p \equiv \alpha_1 + \alpha_2 + \alpha_3 + \cdots \not\equiv 0 \pmod{p}. \end{aligned}$$

The substitutions S', $A^{-1}S'A$, $A^{-2}S'A^2$, $\cdots$, all belonging to Q_1, are readily seen to generate a group of order p^p at least. The group Q_2 is accordingly of order p^{p+1} at least. Theorem III, α), is thus true when $Q_2 = P$.

§ 17. Let $Q_2 < P$. The functions v_1, v_2, $\cdots$, v_p (§ 10) contain p^2 factors y_1, y_2, $\cdots$. We may assume that

$$v_i = y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip} \qquad (i = 1, 2, \cdots, p).$$

We shall say that *the p factors of v_i constitute the set V_i*. The letter B shall designate a substitution belonging to Q_3 but not to Q_2. It permutes the letters v_1, v_2, $\cdots$, v_p in some such order as

$$B: \qquad (v_1 v_2 \cdots v_p).$$

Let R represent a substitution of the group Q_2. It has v_i for a relative invariant. We may assume that it permutes the members of the set V_i in the order $(y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip})$, or some power of this order,

$$R: \qquad (y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip})^n.$$

If $n = p$, i. e., if the functions $y_{(i-1)p+1}$, $\cdots$ are relative invariants for R, we shall say that *R is linear in V_i*. Otherwise we shall say that *R is circular in V_i*. Thus, S is linear in V_1, and A is circular in V_1.

§ 18. Let us now suppose that R is linear in V_i and write

$$R(y_{(i-1)p+j}) = \theta^{\beta_j} y_{(i-1)p+j} \qquad (j = 1, 2, \cdots, p).$$

We shall say that R *is of rank* m *in* V_i *if* $\beta_1 + \theta\beta_2 + \theta^2\beta_3 + \cdots + \theta^{p-1}\beta_p$ *is divisible by* $(1-\theta)^m$, *but by no higher power of* $1-\theta$; $\beta_1, \beta_2, \cdots, \beta_p$ *not being all equal* (*mod* p). *If* $\beta_1 \equiv \beta_2 \equiv \cdots \equiv \beta_p \not\equiv 0$ (*mod* p), *we shall say that* R *is of rank* $p-1$; *if* $\beta_1 \equiv \beta_2 \equiv \cdots \equiv \beta_p \equiv 0$ (*mod* p), *we shall say that* R *is of rank* p.

Let A_1 be any substitution contained in Q_2. Then $A_1 R A_1^{-1} R^{-1}$ is linear in V_i. It is of rank p if A_1 is linear in V_i; if A_1 is circular, the commutator in question is of rank $m+1$ if $m < p$, and of rank p if $m = p$.

§ 19. We can construct by the method of § 16, using S and B, a substitution S_1, $< Q_2$, such that

$$S_1(v_1) = \theta v_1, \qquad S_1(v_i) = v_i \qquad (i = 2, 3, \cdots, p).$$

The following three cases may arise:

(1) S_1 is linear in $V_1, V_2, \cdots, V_p$;
(2) S_1 is circular in $V_1, V_2, \cdots, V_p$;
(3) S_1 is partly linear and partly circular.

Consider case (1). The substitution S_1 is of rank 0 in V_1 and of rank $\geqq 1$ in $V_2, V_3, \cdots, V_p$. The substitution $S_2 = A S_1 A^{-1} S_1^{-1}$ is of rank 1 in V_1 and of rank $\geqq 2$ in $V_2, V_3, \cdots$. The substitution $S_3 = A S_2 A^{-1} S_2^{-1}$ is of rank 2 in V_1 and of rank $\geqq 3$ in $V_2, V_3, \cdots$. Proceeding thus, we get a series of substitutions of which the last, $S_p = A S_{p-1} A^{-1} S_{p-1}^{-1}$, is of rank $p-1$ in V_1 and of rank p in $V_2, V_3, \cdots$.

The substitutions

$$S_p,\ B S_p B^{-1},\ B^2 S_p B^{-2},\ \cdots,\ B^{p-1} S_p B^{-p+1}$$

will generate a group Q' of order p^p at least. This group does not contain S_{p-1}. The substitutions

$$S_{p-1},\ B S_{p-1} B^{-1},\ B^2 S_{p-1} B^{-2},\ \cdots,\ B^{p-1} S_{p-1} B^{-p+1}$$

and the group Q' will generate a group Q'' of order p^{2p} at least, etc. Finally, we have a group $Q^{(p)}$, of order $\geqq p^{p^2}$, linear in $V_1, V_2, \cdots, V_p$, which group with A and B generate a group of order $\geqq p^{p^2+2}$.

§ 20. Consider case (2). Constructing the substitutions

$$R_1 = B S_1 B^{-1}, \qquad R_2 = S_1 R_1 S_1^{-1} R_1^{-1},$$

we find that R_2 is linear in $V_1, V_2, \cdots, V_p$, of rank 1 in V_1 and V_2 and of rank $\geqq 2$ in $V_3, V_4, \cdots, V_p$.

Now, S is linear in V_1 (§ 17). It cannot be linear in all the sets

$V_2, V_3, \cdots$, or S_1 would be linear throughout. Hence there is a substitution, $W = B^n S B^{-n}$, which is circular in V_1 and linear in V_2. Then the substitution $R_3 = W R_2 W^{-1} R_2^{-1}$ is linear in all the sets $V_1, V_2, \cdots$. It is of rank 2 in V_1 and of rank $\geqq 3$ in the remaining sets. Proceeding as in § 19, we construct a group $Q^{(p-2)}$ of order $\geqq p^{(p-2)p}$, linear throughout, a subgroup in the group generated by A, B and R_3. This group $Q^{(p-2)}$ and the substitutions R_2, BR_2B^{-1}, $B^2R_2B^{-2}$, $\cdots$ generate a group $Q^{(p-1)}$ of order $\geqq p^{p^2-p-1}$, linear throughout. This group and the substitutions S_1, BS_1B^{-1}, $B^2S_1B^{-2}$, $\cdots$ generate a group $\leqq Q_2$, of order $\geqq p^{p^2-1}$. Accordingly, the order of Q_3 is $\geqq p^{p^2}$.

The same processes, with slight variations, will dispose of case (3). Theorem III, α), will be found true.

§ 21. In proving Theorem III, β), we may assume that $Q_2 = P$.

There is a substitution $S' < Q_2$, but not $< Q_1$, which can take the place of S in Theorem I, Q_1 being substituted for P_1 in the theorem, and a corresponding group Q' taking the place of Q_1 in Theorems II and III, α). If $P = Q'$, Theorem III, β) is true. If $P > Q'$, then there exists a set of functions

$$V_1' : \quad y_1', y_2', \cdots, y_p',$$

corresponding to the set V_1 defined in § 17. These functions are permuted transitively by P; otherwise the order of this group would be $\geqq p^{p^2}$ [Theorem III, α)]. We will look upon P as transforming simultaneously the two sets V_1 and V_1', making use of the results of § 16.

We may evidently assume $S' = A$ of § 16. *Then A is linear and of rank* 0 *in V_1', and is circular in V_1.*

There must be a substitution A' which is circular in V_1'. We can evidently find an integer n such that the substitution $A'A^n$ is linear in V_1'. Again, such numbers m and k can be found that the substitution

$$R = (A'A^n)^k S^m$$

is linear and of rank 0 *in V_1 and is circular in V_1'.*

The substitutions A and R will now generate a group of order $\geqq p^{2p-1}$. To prove this, we construct first $R_1 = ARA^{-1}R^{-1}$, which is linear and of rank 1 in both V_1 and V_1'; then the substitutions

$$R_2 = AR_1A^{-1}R_1^{-1}, \quad R_3 = AR_2A^{-1}R_2^{-1}, \quad \cdots,$$

which are all linear, of rank p in V_1', and of ranks 2, 3, $\cdots$, respectively, in V_1; finally, we construct the substitutions

$$W_2 = RR_1R^{-1}R_1^{-1}, \quad W_3 = RW_2R^{-1}W_2^{-1}, \quad W_4 = RW_3R^{-1}W_3^{-1}, \quad \cdots,$$

all linear, of rank p in V_1, and of ranks 2, 3, 4, $\cdots$, respectively, in V_1'.

The substitutions $R_1, R_2, R_3, \cdots; W_2, W_3, \cdots$ will generate a group of order $\geqq p^{2p-3}$, linear in V_1 and V_1'. Again this group together with A and R will generate a group of order $\geqq p^{2p-1}$.

§ 22. Theorem IV is proved as follows. The invariant subgroup H_1 exists unless the condition $S(\pi) = \pi$ is satisfied (cf. § 6). But Theorems I, II and III were true in consequence of this condition.

INFINITE DISCONTINUOUS GROUPS OF BIRATIONAL TRANSFORMATIONS WHICH LEAVE CERTAIN SURFACES INVARIANT*

BY

VIRGIL SNYDER

It was shown by SCHWARZ † that no algebraic curve of genus greater than unity can remain invariant under a continuous group of birational transformations. Later HURWITZ ‡ showed that no such curve could belong to any birational group of infinite order.

The corresponding theory for surfaces is by no means complete. While those belonging to continuous groups have been determined, only a few isolated examples are known of surfaces having an infinite discontinuous group. §

All the groups which have been discussed are illustrations of two principles, the first of which refers to quartic surfaces and will be considered in two parts, the latter including the former as a particular case; the second principle is applicable to a much wider category. I propose to discuss these principles and apply the second to the determination of an extended family of new surfaces having an infinite group.

---

* Presented to the Society February 27, 1909.

† *Ueber diejenigen algebraischen Gleichungen zwischen zwei veränderlichen Grössen, welche eine Schaar rationaler eindeutig umkehrbarer Transformationen in sich zulassen*, Crelle's Journal, vol. 87 (1879), pp. 139–146.

‡ *Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen*, Mathematische Annalen, vol. 32 (1888), pp. 290–308, reprinted from the Göttinger Nachrichten of February 7, 1887. See also NOETHER, Mathematische Annalen, vol. 21 (1883).

§ HUMBERT: *Sur la décomposition des fonctions θ en facteurs*, Comptes Rendus, vol. 126, (1898), pp. 394–396; *Sur les fonctions abéliennes singulières*, ibid., pp. 508–510, and Liouville's Journal, ser. 5, vol. 6 (1900), pp. 279-386, see page 372; PAINLEVÉ: *Sur les surfaces qui admettent un groupe infini discontinu de transformations birationnelles*, Comptes Rendus, vol. 126, (1898), pp. 512-514; HUTCHINSON: *The Hessian of the cubic surface II*, Bulletin of the American Mathematical Society, vol. 6 (1900), pp. 328–337, and *On some birational transformations of the Kummer surface into itself*, ibid., vol. 7 (1901), pp. 211-217; FANO: *Sopra alcune superficie del 4° ordine rappresentabili sul piano doppio*, Rendiconti del Istituto Lombardo, vol. 39 (1906), pp. 1071–1086. The first three of these articles are concerned with special forms of the Kummer surface; the treatment is entirely transcendental. The next two treat the general Kummer surface and two others into which it can be transformed; the treatment is partly transcendental and partly algebraic. The last one gives an outline of the theory of those quartic surfaces having a net of hyperelliptic curves; the treatment is purely geometric.

§ 1. *Quartic surfaces that possess a net of hyperelliptic curves of genus two.*

a. The nodal inversion (N).

1. The line joining any point A_1 on a quartic surface F_4 to a conical point N of the surface will meet it again in a point B_1. A $(1, 1)$ correspondence exists between A_1, B_1, defining a non-linear birational transformation of order 2 which will be indicated by (N). If F_4 has two conical points N_1, N_2, these define two such transformations (N_1), (N_2). Every plane section through N_1N_2 will remain invariant under the group generated by (N_1), (N_2). By taking N_1, N_2 as the vertices $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ of the tetrahedron of reference the equation of a plane section may be written in the form

$$x_1^2(ax_2^2+bx_2x_3+cx_3^2)+x_1x_4(a'x_2^2+b'x_2x_3+c'x_3^2)+x_4^2(a''x_2^2+b''x_2x_3+c''x_3^2)=0.$$

The necessary and sufficient condition that the operations (N_1), (N_2) are commutative is the vanishing of the determinant *

$$(a) \qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

If now the binodal quartic be transformed into a cubic, the operations (N_1), (N_2) become ordinary quadric inversions with regard to the polar conics of two ordinary points of the curve. If we still call these points N_1, N_2, the geometric condition that (N_1), (N_2) be commutative is that the tangents at N_1, N_2 meet on the curve, i. e., that N_1, N_2 be conjugate points. In general, the question whether (N_1), (N_2) define a finite group is thus reduced to STEINER'S celebrated "Schliessungsproblem." †

In general, if $(N_1N_2)^k=1$, k must satisfy a certain relation which can be most easily expressed in terms of elliptic functions. Since any binodal quartic or non-singular cubic can be birationally derived from a space quartic curve of the first kind c_4, the $(2, 2)$ correspondence between the points in which the lines through N_1, N_2 meet the curve again can be defined by means of the generators of a quadric surface passing through the quartic curve. The quartic curve may be defined by the equations

$$x_1=\rho\wp''(u), \qquad x_2=\rho\wp'(u), \qquad x_3=\rho\wp(u), \qquad x_4=\rho.$$

Four points corresponding to the values u_1, u_2, u_3, u_4 of the parameter u will lie in the same plane if, and only if,

$$u_1+u_2+u_3+u_4\equiv 0 \pmod{2\omega_1, 2\omega_2}.$$

* STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), p. 267.

† J. STEINER: *Geometrische Lehrsätze*, Crelle's Journal, vol. 32 (1845), pp. 182–184. If the cycle is closed for one point, it will be for every point.

Given any two points u and v on c_4. Through the line joining them can be passed a quadric surface of the pencil having c_4 for basis curve. If $u + v \equiv c$, then the parameters of the points on any generator of the second system must satisfy the relation $u + v \equiv -c$. Starting with the point u, we can obtain v by the equation $u + v \equiv c$, then from v we can find u_1 on the other generator by $v + u_1 \equiv -c$. Similarly $u_1 + v_1 \equiv c$, $v_1 + u_2 \equiv -c$, etc. If $u_k = u$, by eliminating the intermediate terms we obtain the condition

$$2kc \equiv 0.$$

For an arbitrary F_4 with two or more double points this condition will not be satisfied for any finite integer k, hence:

Quartic surfaces having m $(1 < m \leqq 16)$ double points exist which are invariant under a noncontinuous group of birational transformations of infinite order.

2. This result is interesting in view of KLEIN's* question regarding operations which leave the 16-nodal Kummer surface invariant. By means of properly chosen operations of the linear G_{16}, any (N_i) can be transformed into any other (N_k). This gives rise to at least thirty-two operations. By duality we have at once 32 more. The question is whether the surface is invariant under any other than these 64 operations. This question was answered in the affirmative by HUMBERT† by means of hyperelliptic theta functions.

On applying condition (a) to the equation of the surface referred to a Göpel tetrad, this question can be answered in the affirmative immediately. Similarly, by making use of the invariants whose vanishing expresses that $(N_i N_k)$ is finite it is seen that the surface has an infinite group, as the condition for finiteness would impose a relation among the three essential constants of the surface. By means of the known G_{32} the generators of this group can now be easily determined.

3. Another interesting case is furnished by the Weddle surface, the locus of the vertex of a quadric cone through six given points. Since this surface can be birationally transformed into the Kummer surface, it must have a G_{32}. This group was shown by BAKER to be defined by the six operations (N_i).‡

* *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich ein- und umgeschrieben sind*, Mathematische Annalen, vol. 17 (1885), pp. 106–142. See p. 142.

† *Théorie générale des surfaces hyperelliptiques*, Liouville's Journal, ser. 4, vol. 9 (1893), pp. 29–170 and pp. 361–475. See page 466.

‡ This same result can be obtained from the parameters belonging to two points collinear with N_i as given in HUMBERT's third paper cited above, p. 470, but this fact is not there mentioned. BAKER: *Elementary note on the Weddle quartic surface*, Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), pp. 247–261, gives an algebraic proof. A much more extensive discussion of the transformations of both surfaces is given by BAKER: *An introduction to the theory of multiply periodic functions* (1907); see pp. 69–82.

By the preceding principles this theorem can be obtained at once as follows. The equation of the surface may be written in the form

$$\begin{vmatrix} \frac{a_1}{x_1} & x_1 & a_1 & 1 \\ \frac{a_2}{x_2} & x_2 & a_2 & 1 \\ \frac{a_3}{x_3} & x_3 & a_3 & 1 \\ \frac{a_4}{x_4} & x_4 & a_4 & 1 \end{vmatrix} = 0,$$

the six nodes being at the four vertices of the tetrahedron of reference and the two points $(1, 1, 1, 1), (a_1, a_2, a_3, a_4)$. In a plane containing any pair of nodes, as $k_4x_3 = k_3x_4$, the two points on $x_1 = 0$ are $(0, 0, k_3, k_4)$, $[0, k_3(a_2-a_4) + k_4(a_3 - a_2), k_3(a_3 - a_4), k_4(a_3 - a_4)]$. Similarly the two points on $x_2 = 0$ are $(0, 0, k_3, k_4)$, $[k_3(a_1 - a_4) + k_4(a_3 - a_1), 0, k_3(a_3 - a_4), k_4(a_3 - a_4)]$. The necessary and sufficient condition that $(N_1), (N_2)$ be commutative is that the point $[k_3(a_1 - a_4) + k_4(a_3 - a_1), k_3(a_2 - a_4) + k_4(a_3 - a_2), k_3(a_3 - a_4), k_4(a_3 - a_4)]$ lie on the surface. By direct substitution we find that it does lie on the surface; similarly for the other pairs of nodes.

4. Moreover the result can be obtained geometrically. Let c_3 be the space cubic curve passing through the six points N_i. It lies on the Weddle surface defined by these points. The quadric cone having its vertex at N_i and passing through the five other points N_k contains c_3 and is the tangent cone at the conical point N_i of the Weddle surface.*

Any plane section through N_iN_k will contain the line N_iN_k and a cubic curve passing through N_iN_k. The tangent to the plane cubic at N_i is the generator of the tangent cone at N_i lying in the given plane and not passing through N_k. Similarly for N_k. Since these generators meet on c_3, it follows that N_i, N_k are conjugate points with regard to every plane cubic section passing through them, hence (N_i), (N_k) are commutative.

It is known that through any given plane quartic curve a Kummer surface can be passed, while all the plane sections of a Weddle surface are restricted, the relation among the constants being expressed by the equation

$$A^2 + 144B = 0,$$

wherein A is the cubic and B the sextic invariant of a quartic curve.†

* HIERHOLZER: *Ueber Kegelschnitte im Raume*, Mathematische Annalen, vol. 2 (1870), pp. 563-586. See page 582.

† F. MORLEY and J. R. CONNER: *Plane sections of a Weddle surface*, American Journal of Mathematics, vol. 31 (1909), pp. 263-270.

The preceding results furnish a new geometric interpretation of this important theorem for the case in which the plane passes through two conical points of the surface.

§ 1*b*. *The Involutions* (I).

5. A quartic surface F_4 of genus one ($p_a = p_g = p = 1$) which contains one curve of genus 2 will contain a net of such curves, any two of which will meet in two points, forming a rational involution I. (FANO, l. c.) The curves of order 4 are plane sections, and the surface has a double point. These curves define the inversions (N). In case the curves are of order 5 they are cut from F_4 by a net of quadrics, all having for residual the same twisted cubic.

Given a curve c_6 of order six and genus 2 on F_4. It is then one of a net σ_1, defining an involution I_1. Each c_6 of σ_1 will be cut in ∞^1 pairs of points of I_1, forming the canonical g_2^1. Through each c_6 pass ∞^2 cubic surfaces F_3. The residual intersection will be another sextic c_6', also of genus 2, and belonging to a net σ_2, distinct from σ_1 provided no F_3 of the net can be found which touches F_4 in every point of c_6. The net σ_2 will define an involution I_2 on F_4.

The ∞^1 lines determined by the pairs of points in g_2^1 on c_6 define a ruled cubic surface R_3 whose double directrix is a quadrisecant of c_6. The other points of intersection of each generator with F_4 will belong to the canonical g_2^1 on the residual c_6'. The curves c_6, c_6' have four points of intersection on the double directrix. Every line joining a pair of points in I_1 will also join a pair of points in I_2. The lines joining any point to its conjugate will therefore define a congruence of order 2, one of the lines joining a given point to its conjugate in I_1 and the other joining the same point to its conjugate in I_2. Starting with any point on the surface we can first find its conjugate in I_1 by a birational transformation of order 2, then the conjugate of the latter as to I_2, also a transformation of order 2. If a line of I_1 be given, there are two lines, one through each of the points of this line belonging to I_1, which connect it with its conjugate in I_2. Hence between the lines of I_1, I_2 there is a (2, 2) correspondence, and the condition for periodicity is reduced to that of the preceding case.*

The determination of these hyperelliptic curves by algebraic processes is in general a very difficult problem.

§ 2. *Systems of bitangents* (T).

6. Let $a \equiv \Sigma a_i x_i = 0$, $b = 0$ be the equations of a straight line, the coefficients a_i, b_i being rational functions of two non-homogeneous parameters κ, τ. If the coördinates of a fixed point y be substituted for x_i, the number of roots in κ, τ defines the order of the congruence T, that is, the number of lines

* FANO, l. c., showed by a different method that the operation $(I_1 I_2)$ is in general of infinite order.

belonging to the system which pass through the point. The locus of the point y for which two of the lines coincide is the focal surface of the congruence. Thus, except for particular cases, the line of a given congruence of any order which touches the focal surface F at a given point P can be rationally separated from the other lines passing through P by the process of partial elimination. We shall be concerned only with such congruences as have a single line (counted twice) passing through P on F, and lying in its tangent plane. The other lines of T passing through P have one point of intersection with F at P.*

Congruences of the first order can have no focal surface; for those of the second order there is no residual line, while in the case of cubic congruences points on the focal surface are characterized by one double and one single root in κ, τ.

Every line of T is a bitangent to F, the points of tangency being P_1, P_2. The operation of interchanging P_1, P_2 is a birational transformation of order 2 which will be denoted by (T). If F is the complete focal surface for more than one congruence, we have two or more operations of order 2, and their product will be in general of infinite order. The necessary and sufficient condition that the transformations (T_1), (T_2) be birational is that the two congruences T_1, T_2 be rationally separable.

7. A number of important surfaces having this property can be obtained as the apparent contour of SEGRE's cubic variety Γ in four dimensional space S_4,† quartics by projecting from a point on Γ, and sextics by projecting from any point in S_4. In particular, if Γ contains a plane and can be generated by trilinear systems, the contour will be the focal surface of at least two congruences, and the equations of the transformations (T_1), (T_2) can be determined.

Let

$$\Gamma \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0$$

be the equation of a cubic variety containing the plane $x_4 = 0$, $x_5 = 0$. A space $\kappa_2 x_4 - \kappa_1 x_5 = 0$ passing through this plane will cut from Γ a quadric surface, one such quadric passing through any point of Γ.

The two systems of generators are rationally separable, being

$$\frac{u_3}{u_1 k_2 - u_2 k_1} = \frac{x_3}{x_1 k_2 - x_2 k_1} = \tau_1, \qquad \frac{u_3}{x_3} = \frac{u_1 k_2 - u_2 k_1}{x_1 k_2 - x_2 k_1} = \tau_2,$$

* An example of congruences of another kind is furnished by a family of quadrics having one variable parameter, when the equations of the two systems of generators cannot be rationally separated.

† *Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario.* Memorie di Torino, ser. 2, vol. 39 (1889), pp. 1–48.

the second one being defined by the pencil of spaces passing through the other plane $u_3 = 0$, $x_3 = 0$ on Γ.

If now a plane be passed through any line τ_1 and a fixed point K, the section of Γ made by this plane will consist of τ_1 and a conic cutting τ_1 in two points P', P''. The locus of the points P_1, P_2 in which the lines KP', KP'' are cut by an S_3 not passing through K is the focal surface of the congruence T_1, the points P_1, P_2 being the points of contact. If $k \equiv (c_1, c_2, \cdots)$, and S_3 be defined by $\Sigma e_i x_i = 0$, the equations take the form

$$\frac{u_3 - \tau_1(u_1 k_2 - u_2 k_1)}{u_3(c) - \tau_1[u_1(c)k_2 - u_2(c)k_1]} = \frac{x_3 - \tau_1(x_1 k_2 - x_2 k_1)}{c_3 - \tau_1(c_1 k_2 - c_2 k_1)} = \frac{k_2 x_4 - k_1 x_5}{k_2 c_4 - k_1 c_5}$$

from which x_5, for example, can be eliminated by means of $\Sigma e_i x_i = 0$. If τ_1 be eliminated, a cubic in $k_1 : k_2$ results, whose envelope is the focal surface.

If then we start with a point $(x_1', \cdots, x_5')$ or x' on Γ, the associated S_3 and the line τ_1 are known and the points $P' \equiv (x')$, $P'' \equiv (x'')$ are at once defined as the roots of a quadratic equation, one of which is known. The other root is expressed as a rational function of the given one, and the equations of (T_1) are determined. In the same way we determine (T_2).

If the system T_1 belongs to a linear complex, the points P_1, P_2 are poles of the tangent planes at P_2, P_1 respectively, so that (T_1) may be regarded as the product of the two commutative operations, duality as to the surface and duality as to the complex. When T_1, T_2 both belong to linear complexes, the group generated by (T_1), (T_2) may be finite. In particular, the necessary and sufficient condition that (T_1), (T_2) are commutative is that the complexes to which T_1, T_2 belong are in involution.

The lines of the congruence T_1 can be arranged on a system of ∞^1 quadric surfaces, the congruence T_2 being composed of the other system of generators of these quadrics. Any line l of T_1 will therefore determine two lines m_1, m_2 of T_2, so that a (2, 2) correspondence exists between l, m. Thus the locus of P_1, P_2 corresponding to the quadric defined by l, m is a space curve c_4 of order 4 of the first kind. By the operations (T_1), (T_2) the curve c_4 remains invariant, so that each quadric of the system goes into itself. The points P_1, P_2 define a (2, 2) correspondence upon c_4, hence the discussion of the periodicity of the operation $(T_1 T_2)$ can be reduced to that of the preceding case.*

When the focal surface is of order 6 the residual section made by any quadric of bitangents is another c_4 which is also the curve of contact of another quadric. If the line $P_1 P_2$ cuts F_6 in Q_1, Q_2 the operation of interchanging Q_1, Q_2 is also birational, since only a single line of each congruence passes through Q_1

* For the literature concerning the (2, 2) correspondence on c_4, see O. STAUDE: *Flächen zweiten Grades und ihre Systeme und Durchdringungskurven*, Encyklopädie der mathematischen Wissenschaften, III C2., no. 123; in particular, footnote 513.

apart from the one (counted twice) which touches F_6 at Q_1. All the theorems regarding periodicity can be applied immediately to this case also.

8. An interesting illustration is furnished by the variety Γ having ten nodes. The contour from any point on Γ is the general Kummer surface. The six systems of bitangents belong to six linear complexes mutually in involution, hence the operations (T_i) generate the well-known G_{32} first found by Klein.*

The product of any two operations (T_i), (T_k) is a linear transformation, and thus the linear group of order 16 is defined. Many of the properties of the Kummer surface and of its systems of bitangents can be deduced readily from this starting point.

9. An illustration of a different kind, wherein the group generated by the $(T_i)(T_k)$ is finite, is obtained from the variety

$$\Gamma \equiv x_1x_2x_3 + \lambda x_4x_5x_6 = 0, \qquad \Sigma x_i = 1, \qquad \lambda \neq 1, \tag{1}$$

the center of projection being $P \equiv (1, 1, 1, -1, -1, -1)$. The details of this case will sufficiently illustrate the general procedure. The line of the system† I which passes through the point x' may be expressed by the equations

$$\frac{x_1'x_4 - x_4'x_1}{x_1' + x_4'} = \frac{x_2'x_5 - x_5'x_2}{x_2' + x_5'} = \frac{x_3'x_6 - x_6'x_3}{x_3' + x_6'} = l. \tag{2}$$

Solve these equations for x_4, x_5, x_6 in terms of l and substitute the results in (1). The equations of the conic and of the line may be written in the form

$$x_5'x_6'(x_1' + x_4')x_2x_3 + x_4'x_6'(x_2' + x_5')x_1x_3 + x_4'x_5'(x_3' + x_6')x_1x_2 = 0, \tag{3}$$

$$x_2'x_3'(x_1' + x_4')x_1 + x_1'x_3'(x_2' + x_5')x_2 + x_1'x_2'(x_3' + x_6')x_3 = 0. \tag{4}$$

If now x' be taken on the first polar of Γ as to P,

$$H \equiv x_1x_2 + x_1x_3 + x_2x_3 - \lambda(x_4x_5 + x_5x_6 + x_6x_4) = 0, \tag{5}$$

it will be one of the points of intersection of the line and the conic. Making (3), (4) simultaneous and making use of (2), we obtain the coördinates of x'', the second point of intersection. The results are

$$x_1'' = \frac{1}{x_4'}, \quad x_2'' = \frac{1}{x_5'}, \quad x_3'' = \frac{1}{x_6'}, \quad x_4'' = \frac{1}{x_1'}, \quad x_5'' = \frac{1}{x_2'}, \quad x_6'' = \frac{1}{x_3'}. \tag{6}$$

* *Zur Theorie der Liniencomplexe des ersten und zweiten Grades*, Mathematische Annalen, vol. 2 (1870), pp. 201–226.

† For this notation and the discussion of this variety, see my paper: *Surfaces derived from the cubic variety having nine double points in four dimensional space*, these Transactions, vol. 10 (1909), pp. 71–78.

Let the lines joining x' to P be cut by the polar S_3 of P as to Γ

$$x_1 + x_2 + x_3 + \lambda(x_4 + x_5 + x_6) = 0. \tag{7}$$

To distinguish between points x on Γ and points on the S_3 defined by (7), coördinates in the latter will be denoted by y_i. It is defined by

$$y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0. \tag{8}$$

From a point x on $\Gamma(x)$ we obtain the corresponding point in S_3 by means of the equations

$$\begin{aligned} y_1 &= x_2 + x_3 - 2x_1, & y_4 &= x_5 + x_6 - 2x_4, \\ y_2 &= x_1 + x_3 - 2x_2, & y_5 &= x_4 + x_6 - 2x_5, \\ y_3 &= x_1 + x_2 - 2x_3, & y_6 &= x_4 + x_5 - 2x_6, \end{aligned} \tag{9}$$

and the reciprocal relations, giving x when y is known, are

$$\begin{aligned} x_1 &= 2y_1 H(y) - 3\Gamma(y), & x_4 &= 2y_4 H(y) + 3\Gamma(y), \\ x_2 &= 2y_2 H(y) - 3\Gamma(y), & x_5 &= 2y_5 H(y) + 3\Gamma(y), \\ x_3 &= 2y_3 H(y) - 3\Gamma(y), & x_6 &= 2y_6 H(y) + 3\Gamma(y). \end{aligned} \tag{10}$$

Now by means of equations (6), (9), (10) we can obtain the equations of the birational transformation in S_3. The results are

$$\begin{aligned} \rho y_1' &= -\lambda[2H(y)(y_4y_5 + y_5y_6 - 2y_4y_6) + 3\Gamma(y)(y_5 - 2y_4 - 2y_6)], \\ \rho y_2' &= -\lambda[2H(y)(y_4y_6 + y_5y_6 - 2y_4y_5) + 3\Gamma(y)(y_6 - 2y_4 - 2y_5)], \\ \rho y_3' &= -\lambda[2H(y)(y_4y_6 + y_4y_5 - 2y_5y_6) + 3\Gamma(y)(y_4 - 2y_5 - 2y_6)], \\ \rho y_4' &= 2H(y)(y_1y_2 + y_2y_3 - 2y_1y_3) - 3\Gamma(y)(y_2 - 2y_1 - 2y_3), \\ \rho y_5' &= 2H(y)(y_1y_3 + y_2y_3 - 2y_1y_2) - 3\Gamma(y)(y_3 - 2y_1 - 2y_2), \\ \rho y_6' &= 2H(y)(y_1y_2 + y_1y_3 - 2y_2y_3) - 3\Gamma(y)(y_1 - 2y_2 - 2y_3), \end{aligned} \tag{11}$$

the equations of the surface being

$$y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0, \qquad 27(1-\lambda)\Gamma^2(y) + 4H^3(y) = 0.$$

The transformation defined by (11) is not a Cremona transformation, being birational for points of the surface only.

In the same manner we may obtain the systems (T_2), (T_3), defined by II, III, ···. If we use the notation $(x_ix_kx_l)$ to indicate the cyclic substitution

$x_i = \rho x_k'$, etc., the results may be written in the form

$$T_2 = T_1 l_2,\ T_3 = T_1 l_2^2,\ T_4 = T_1 l_4,\ T_5 = T_1 l_5,\ T_6 = T_1 l_4 l_5 l_4,$$

wherein

$$l_2 = (y_1 y_2 y_3)(y_4 y_6 y_5);\ l_4 = (y_2 y_3)(y_5 y_6);\ l_5 = (y_1 y_2)(y_4 y_5).$$

Moreover

$$T_1 l_2 = l_2 T_1,\ T_1 l_4 = l_4 T_1,\ T_1 l_5 = l_5 T_1,$$

from which it follows that $T_1, \cdots, T_6$ generate a group of order 36 which contains a linear subgroup of order 18. The surface is also invariant under the odd substitutions $t = (y_2 y_3)$ etc., making another linear group of order 18. The operations (T_i) combine with these in the same manner as with the preceding, making a total group of order 72.

10. From any point P_1 on F_6 can be drawn just one line of I touching F_6 at P_1. Let P_2 be the second point of contact of this line. Similarly, the line $P_2 P_3$ belongs to IV and touches F_6 at P_2 and at P_3. The line $P_3 P_4$ belongs to I. Since $(T_1 T_4)^2 = 1$ it follows that $P_4 P_1$ belongs to IV and has its points of contact at P_4 and at P_1. The vertices P_1, P_2, P_3, P_4 define a tetrahedron inscribed in F_6 and the planes $P_1 P_2 P_3$, etc. are all tangent planes. The transformation $(T_1 T_4)$ transforms P_1 into P_3 and P_2 into P_4; it is the axial involution l_4. We have the following theorem:

There are nine systems of ∞^2 tetrahedra which are inscribed in and circumscribed about F_6. Two opposite edges of the tetrahedra of each system always meet two fixed lines.

11. When the line σ touches the residual conic, section of Γ, $P_1 = P_2$ and the corresponding line in S_3 has four coincident points in common with F_6. The locus of the point of contact in I is a curve defined as the intersection of $\Gamma(x)$ with $H(x)$ and the variety defined by the equation

$$x_1 x_2 x_3 x_4 x_5 x_6 \left[\frac{(x_1 + x_4)^4}{x_1^2 x_4^2} + \frac{(x_2 + x_5)^4}{x_2^2 x_5^2} + \frac{(x_3 + x_6)^4}{x_3^2 x_6^2} \right]$$

$$-2(x_1+x_4)^2(x_2+x_5)^2(x_3+x_6)^2 \left[\frac{x_1 x_4}{(x_1 + x_4)^2} + \frac{x_2 x_5}{(x_2 + x_5)^2} + \frac{x_3 x_6}{(x_3 + x_6)^2} \right] = 0.$$

By means of equations (6), (9), (10) the equations of the curve in S_3 can be obtained. Let C_1 denote the curve at the points of which the lines of I have four point contact. From any point P_1 of this curve draw the line of IV. From the preceding theorem the second point of contact must also lie on C_1. The ruled surfaces belonging to IV, V, VI which have C_1 for directrix curve are such that every generator of each is a bisecant of the curve. If the line of II touching F_6 at P_1 also touches it at P', and the line $P' P''$ belongs to I, then P'' is a point of C_2. *The six curves C_i are birationally equivalent.*

CORNELL UNIVERSITY,
February, 1909.

PROPER MULTIPLE INTEGRALS OVER ITERABLE FIELDS

BY

ERNEST B. LYTLE*

Introduction.

The theory of multiple integrals has generally been confined to functions having metric (defined later in § 1) domains of definition. In his book † and in an article on *Improper Multiple Integrals* ‡ PIERPONT has given a definition of integrals applicable to non-metric domains.

Using this PIERPONT definition of an integral I have found a new class of fields for which the fundamental relation

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{G}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} f \leqq \bar{\int}_{\mathfrak{A}} f$$

holds. This general class of fields, which I have called iterable, includes all metric fields and a large class of non-metric fields.

For simplicity the results are stated throughout the paper for functions of two independent variables; but the method of treatment is general and the extension to the case of any number of independent variables is thought to be obvious. The notation used by PIERPONT has been followed, and the reader is referred for further explanation to his *Lectures*.

§ 1. *Preliminary definitions.*

Let $\mathfrak{A} = \{x, y\}$ be a two-dimensional point aggregate lying within a square. Effect a rectangular division of the plane of norm δ. Let

$$\delta_1', \ \delta_2', \ \delta_3', \ \cdots$$

be those rectangles containing at least one point of $\mathfrak{A}$. Let

$$\delta_1'', \ \delta_2'', \ \delta_3'', \ \cdots$$

denote those rectangles all of whose points belong to $\mathfrak{A}$. Then

$$\lim_{\delta=0} \Sigma \delta_i' = \bar{\mathfrak{A}}$$

* Presented to the Society (New York), April 25, 1908.

† *Lectures on the Theory of Functions of Real Variables*, vol. 1 (1905). I shall refer to this as *Lectures*.

‡ These Transactions, vol. 7 (1906). Referred to later under the title *Improper Integrals*.

is defined to be the *outer* (upper) *content* of $\mathfrak{A}$, and

$$\lim_{\delta=0} \Sigma\delta_i'' = \underline{\mathfrak{A}}$$

is the *inner* (lower) *content* of $\mathfrak{A}$. It has been shown (*Lectures*, p. 513) that these limits always exist and are finite. When

$$\bar{\mathfrak{A}} = \underline{\mathfrak{A}}$$

their common value is called the *content* of $\mathfrak{A}$, and $\mathfrak{A}$ is said to be a *metric aggregate.*

Let $\mathfrak{A} = \{x, y\}$ be a limited point aggregate. Let $f(x, y)$ be a limited single-valued function defined over $\mathfrak{A}$. Effect a rectangular division D of the plane of norm δ. Let

$$\delta_1', \ \delta_2', \ \delta_3', \ \cdots,$$

be those rectangles of D which contain points of $\mathfrak{A}$. Let M_i, m_i be the maximum and minimum respectively of $f(x, y)$ in the rectangle δ_i'. Put

$$\bar{S}_D = \Sigma M_i\delta_i', \qquad \underline{S}_D = \Sigma m_i\delta_i'. \tag{1}$$

Then

$$\lim_{\delta=0} \bar{S}_D = \bar{\int}_{\mathfrak{A}} f \tag{2}$$

is defined to be the *outer upper integral* of $f(x, y)$ over the field $\mathfrak{A}$, and

$$\lim_{\delta=0} \underline{S}_D = \underline{\int}_{\mathfrak{A}} f \tag{3}$$

is the *outer lower integral* of $f(x, y)$ over $\mathfrak{A}$. These limits always exist and are finite. (See *Lectures*, p. 510.)

When the integrals (2) and (3) are equal we denote their common value by

$$\int_{\mathfrak{A}} f \tag{4}$$

and say that f is *integrable* over $\mathfrak{A}$.

It is important to note that the relations

$$\bar{\int}_{\mathfrak{A}} f = \min \bar{S}_D, \qquad \underline{\int}_{\mathfrak{A}} f = \max \underline{S}_D \tag{5}$$

which hold for Riemann integrals are not always true for outer integrals. (See *Lectures*, p. 508.)

Let $\mathfrak{A}$ be a limited plane point set. Let $\mathfrak{C}$ be the points of $\mathfrak{A}$ lying on a line parallel to the y-axis. Let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ on the x-axis. Then when

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}} \tag{6}$$

we define $\mathfrak{A}$ to be *iterable relatively to* x.

If C is a linear section of $\mathfrak{A}$ parallel to the x-axis and B is the projection of $\mathfrak{A}$ on the y-axis, then when

$$\int_B \bar{C} = \bar{\mathfrak{A}} \tag{7}$$

$\mathfrak{A}$ is *iterable relatively to* y.

The following examples show that both iterable and non-iterable aggregates exist.

EXAMPLE 1. Let $\mathfrak{A} = \{x, y\}$ where x takes the rational values in a unit interval and y takes all real values in a unit interval. Then $\mathfrak{A}$ is non-metric, for

$$\underline{\mathfrak{A}} = 0, \qquad \bar{\mathfrak{A}} = 1.$$

But $\mathfrak{A}$ is iterable relative to both x and y, for

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \int_B \bar{C} = 1 = \bar{\mathfrak{A}}.$$

EXAMPLE 2. Let $\mathfrak{A} = \{x, y\}$ be the points of a unit square, specified for rational x by $0 \leqq y \leqq \frac{1}{2}$, and for irrational x by $\frac{1}{2} \leqq y \leqq 1$. Here $\mathfrak{A}$ is non-metric, for

$$\underline{\mathfrak{A}} = 0, \qquad \bar{\mathfrak{A}} = 1.$$

Also $\mathfrak{A}$ is iterable relatively to y, for

$$\int_B \bar{C} = 1 = \bar{\mathfrak{A}},$$

but not iterable relatively to x, for

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \tfrac{1}{2} \neq \bar{\mathfrak{A}}.$$

By similar methods fields may readily be constructed which are iterable neither relatively to x nor relatively to y.

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

and for each x let

$$\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2.$$

Then we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *thoroughly mixed* partials of $\mathfrak{A}$.

The notation $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is used to specify that $\mathfrak{B}$ is the projection of $\mathfrak{A}$ on the x-axis and $\mathfrak{C}$ is a linear section of $\mathfrak{A}$ parallel to the y-axis.

§ 2. *Properties of iterable fields.*

THEOREM 1. *Every metric aggregate is iterable relatively to both x and y.*

It has been shown (see *Lectures*, p. 538) that, for a function $f(x, y)$ limited

in the metric field $\mathfrak{A}$,

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int} dx \bar{\int} f dy \leqq \bar{\int} dx \bar{\int} f dy \leqq \bar{\int}_{\mathfrak{A}} f,$$

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int} dy \bar{\int} f dx \leqq \bar{\int} dy \bar{\int} f dx \leqq \bar{\int}_{\mathfrak{A}} f.$$

In these relations put $f(x, y) = 1$ and they become

$$\bar{\mathfrak{A}} \leqq \underline{\int}_{B} \bar{C} \leqq \bar{\int}_{B} \bar{C} \leqq \bar{\mathfrak{A}}, \qquad \bar{\mathfrak{A}} \leqq \underline{\int}_{\mathfrak{E}} \bar{\mathfrak{C}} \leqq \bar{\int}_{\mathfrak{E}} \bar{\mathfrak{C}} \leqq \bar{\mathfrak{A}}.$$

Hence

$$\int_{\mathfrak{E}} \bar{\mathfrak{C}} = \int_{B} \bar{C} = \bar{\mathfrak{A}},$$

that is, $\mathfrak{A}$ is iterable relatively to x and relatively to y.

LEMMA. *Let $\mathfrak{A}$ be iterable relatively to x. Then if $\delta = \alpha\ \beta$ is a cell of any rectangular division D, we have*

$$\lim_{\delta=0} \Sigma(\alpha\Sigma\beta) = \lim_{\alpha=0} \Sigma(\alpha \lim_{\beta=0} \Sigma\beta).$$

For each x the inequality

$$\lim_{\beta=0} \Sigma\beta \leqq \Sigma_{\beta}$$

will hold. Hence

$$\Sigma(\alpha \lim_{\beta=0} \Sigma\beta) \leqq \Sigma(\alpha\Sigma\beta) \leqq \Sigma\delta.$$

But

$$\lim_{\delta=0} \Sigma(\alpha \lim_{\beta=0} \Sigma\beta) = \lim_{\alpha=0} \Sigma(\alpha \lim_{\beta=0} \Sigma\beta) = \int_{\mathfrak{E}} \bar{\mathfrak{C}},$$

$$\lim_{\delta=0} \Sigma\delta = \bar{\mathfrak{A}}.$$

Since $\mathfrak{A}$ is iterable relatively to x

$$\int_{\mathfrak{E}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}},$$

and our theorem follows.

This lemma enables us to evaluate an iterated limit by a simpler single limit. A similar lemma is true for a field iterable relatively to y.

THEOREM 2. *Every partial unmixed aggregate of an iterable aggregate is iterable.*

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

where $\mathfrak{A}$ is iterable and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed.*

* See *Lectures*, § 711, p. 519.

Then for each x

$$\bar{\mathfrak{C}} \leqq \bar{\mathfrak{C}}_1 + \bar{\mathfrak{C}}_2,$$

and for every x-interval

$$\min \bar{\mathfrak{C}} \leqq \min \bar{\mathfrak{C}}_1 + \max \bar{\mathfrak{C}}_2.$$

Hence by integration over $\mathfrak{B}$,

$$\underline{\int} \bar{\mathfrak{C}} \leqq \underline{\int} \bar{\mathfrak{C}}_1 + \bar{\int} \bar{\mathfrak{C}}_2.$$

But

$$\underline{\int} \bar{\mathfrak{C}}_1 \leqq \bar{\mathfrak{A}}_1, \qquad \bar{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_2.$$

Hence

$$\underline{\int} \bar{\mathfrak{C}} \leqq \underline{\int} \bar{\mathfrak{C}}_1 + \bar{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2. \tag{1}$$

Since $\mathfrak{A}$ is iterable by hypothesis

$$\int \bar{\mathfrak{C}} = \bar{\mathfrak{A}},$$

and since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed

$$\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2.$$

Therefore by (1)

$$\underline{\int} \bar{\mathfrak{C}}_1 + \bar{\int} \bar{\mathfrak{C}}_2 = \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2,$$

or

$$\left(\bar{\mathfrak{A}}_1 - \underline{\int} \mathfrak{C}_1\right) + \left(\bar{\mathfrak{A}}_2 - \bar{\int} \mathfrak{C}_2\right) = 0.$$

But

$$\bar{\mathfrak{A}}_1 - \underline{\int} \bar{\mathfrak{C}}_1 \geqq 0, \qquad \bar{\mathfrak{A}}_2 - \bar{\int} \bar{\mathfrak{C}}_2 \geqq 0,$$

hence

$$\bar{\mathfrak{A}}_1 = \underline{\int} \bar{\mathfrak{C}}_1, \qquad \bar{\mathfrak{A}}_2 = \bar{\int} \bar{\mathfrak{C}}_2. \tag{2}$$

Similarly

$$\underline{\int} \bar{\mathfrak{C}} \leqq \bar{\int} \bar{\mathfrak{C}}_1 + \underline{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2 = \bar{\mathfrak{A}}, \tag{3}$$

from which follows

$$\bar{\mathfrak{A}}_1 = \bar{\int} \bar{\mathfrak{C}}_1, \qquad \underline{\int} \bar{\mathfrak{A}}_2 = \bar{\mathfrak{C}}_2. \tag{4}$$

By (2) and (4), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are each iterable.

COROLLARY. *If a discrete aggregate be subtracted from an iterable aggregate, then the remaining aggregate is iterable.*

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

where $\mathfrak{A}$ is iterable and $\mathfrak{A}_2$ is discrete. Since $\mathfrak{A}_2$ is discrete $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed. Therefore $\mathfrak{A}_1$ is iterable by the preceding theorem.

§ 3. *Proper Integrals.*

A *proper* integral is one in which both the function and its field of integration are limited.

THEOREM 3.* *Let $f(x, y) \geqq 0$ be limited over the limited field $\mathfrak{A}$. Then*

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

If the sign of f is changed, the theorem is converted into the following:

THEOREM 4. *Let $f(x, y) \leqq 0$ be limited over the limited field $\mathfrak{A}$. Then*

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{G}} f.$$

THEOREM 5. *Let $f(x, y)$ and $g(x, y)$ each be limited over the limited field $\mathfrak{A}$. Let them be equal except at a discrete set of points. Then*

(*a*)
$$\bar{\int}_{\mathfrak{A}} f = \bar{\int}_{\mathfrak{A}} g,$$

(*b*)
$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} f = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} g.$$

Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$, where $\mathfrak{A}_2$ is discrete. Let $f = g$ over $\mathfrak{A}_1$. Let D be a rectangular division of the plane of norm $\delta = \alpha \cdot \beta$. Let

$$\delta_{11},\ \delta_{12},\ \delta_{13},\ \cdots$$

be those rectangles which contain no points of $\mathfrak{A}_2$. Let

$$\delta_{21},\ \delta_{22},\ \delta_{23},\ \cdots$$

contain points of $\mathfrak{A}_2$. Then

$$\Sigma M_i \delta_i = \Sigma M_{1i} \delta_{1i} + \Sigma M_{2i} \delta_{2i},$$

$$\Sigma N_i \delta_i = \Sigma N_{1i} \delta_{1i} + \Sigma N_{2i} \delta_{2i},$$

* Restated for later use. For proof see RICHARDSON, these Transactions, vol. 9 (1908), p. 347.

where M_i and N_i are the maxima of f and g respectively in δ_i. Subtracting we get

$$\Sigma M_i \delta_i - \Sigma N_i \delta_i = \Sigma M_{2i} \delta_{2i} - \Sigma N_{2i} \delta_{2i},$$

since $M_{1i} = N_{1i}$. On passing to the limit as $\delta \doteq 0$, this becomes

$$\bar{\int}_{\mathfrak{A}} f - \bar{\int}_{\mathfrak{A}} g = 0,$$

for each limit in the second member is zero because $\mathfrak{A}_2$ is discrete and both f and g are limited. This gives our relation (a).

From (a) and theorem 10 below it follows that

$$\bar{\int}_{\mathfrak{C}} g = \bar{\int}_{\mathfrak{C}} f$$

at all points of $\mathfrak{B}$ except a discrete set. Relation (b) then follows by application of relation (a) to the set $\mathfrak{B}$.

THEOREM 6. *Let $f(x, y)$ be limited over a limited field $\mathfrak{A}$ and negative only at a discrete set of points Δ. Then*

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

Let $g(x, y) = f(x, y)$ at all points of $\mathfrak{A}$ except at Δ where $g = 0$. Then $g \geqq 0$, and by theorem 3

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} g \leqq \bar{\int}_{\mathfrak{A}} g.$$

But by the preceding theorem

$$\bar{\int}_{\mathfrak{A}} g = \bar{\int}_{\mathfrak{A}} f, \qquad \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} g = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f,$$

and substituting these in the above relation we have our theorem.

THEOREM 7. *Let $f(x, y)$ be limited over a limited field $\mathfrak{A}$ and positive only at a discrete set of points. Then*

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f.$$

THEOREM 8. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$. For each x let $\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2$. Let $f \geqq 0$ over $\mathfrak{A}_1$ and $f \leqq 0$ over $\mathfrak{A}_2$. Then*

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

This inequality will appear frequently in the succeeding pages and we shall refer to it as *relation* (A).

Since $\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2$, the positive and negative values of the function are each everywhere dense in $\mathfrak{A}$. Hence every rectangle containing a point of $\mathfrak{A}$ also contains a point of $\mathfrak{A}_1$, and

$$\bar{\int}_{\mathfrak{A}} f = \lim \Sigma_{\mathfrak{A}} M_i \delta_i = \lim \Sigma_{\mathfrak{A}_1} M_i \delta_i = \bar{\int}_{\mathfrak{A}_1} f.$$

Similarly for each x we have

$$\bar{\int}_{\mathfrak{C}} f = \bar{\int}_{\mathfrak{C}_1} f.$$

But the projections of $\mathfrak{A}$ and $\mathfrak{A}_1$ on the x-axis are equal, hence

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_1} f.$$

Since $f \geqq 0$ over $\mathfrak{A}_1$, we have by theorem 3,

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}_1} \qquad \bar{\int}_{\mathfrak{A}_1} f$$

and the equals of both members give

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f. \tag{1}$$

In the same way, using theorem 4 and $\mathfrak{A}_2$, we get

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f. \tag{2}$$

But we always have

$$\underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f. \tag{3}$$

Combining (1), (2) and (3) we get relation (A), which was to be proved.

Examples of thoroughly mixed sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are well known; for example, in $\mathfrak{A}_1$ let x be real and y be rational, and in $\mathfrak{A}_2$ let x be real and y be irrational.

THEOREM 9. *Let $f(x, y)$ be limited over the limited iterable field $\mathfrak{A}$. Then the existence of the multiple integral $\int_{\mathfrak{A}} f$ and the iterated integral $\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$ is sufficient for their equality.*

Let $|f| \leqq F$. Then $f + F \geqq 0$, and by theorem 3

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} (f + F) \leqq \bar{\int}_{\mathfrak{A}} (f + F).$$

But

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f + F\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \int_{\mathfrak{B}}\int_{\mathfrak{C}} (f+F) \leqq \int_{\mathfrak{B}}^{\bar{}}\int_{\mathfrak{C}}^{\bar{}} (f+F) \leqq \int_{\mathfrak{A}}^{\bar{}} (f+F)$$

$$= \int_{\mathfrak{A}}^{\bar{}} f + F\bar{\mathfrak{A}} = \int_{\mathfrak{A}} f + F\bar{\mathfrak{A}}.$$

Hence

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f \leqq \int_{\mathfrak{A}} f + F\left(\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}}\right). \tag{1}$$

Since $f - F \leqq 0$, we get in a similar way by theorem 4,

$$\int_{\mathfrak{A}} f - F\bar{\mathfrak{A}} = \int_{\underline{\mathfrak{A}}} (f-F) \leqq \int_{\mathfrak{B}}\int_{\mathfrak{C}} (f-F) = \int_{\mathfrak{B}}\int_{\mathfrak{C}} f - F\int_{\mathfrak{B}} \bar{\mathfrak{C}},$$

and therefore

$$-F\left(\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}}\right) + \int_{\mathfrak{A}} f \leqq \int_{\mathfrak{B}}\int_{\mathfrak{C}} f. \tag{2}$$

But since $\mathfrak{A}$ is iterable

$$\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}} = 0,$$

and this with (1) and (2) gives

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}}\int_{\mathfrak{C}} f.$$

We observe in this connection that the mere existence of the multiple and the iterated integrals is not sufficient in general for their equality.* The theorem just cited may not be true when the field is not iterable and the function limited, as is shown by the following example.

EXAMPLE 3. Let $f(x, y) = 1$ over the field $\mathfrak{A} = \{x, y\}$ where for rational x, $0 \leqq y \leqq \frac{1}{2}$ and for irrational x, $\frac{1}{2} \leqq y \leqq 1$. Then

$$\int_{\mathfrak{A}} f = 1, \qquad \int_{\mathfrak{B}}\int_{\mathfrak{C}} f = \tfrac{1}{2}.$$

That is, both integrals exist but are unequal.

THEOREM 10. *Let $\mathfrak{A} = \mathfrak{B}\cdot\mathfrak{C}$ be discrete. Then the points $\mathfrak{B}_\sigma$ at which $\bar{\mathfrak{C}} \geqq \sigma$, where σ denotes an arbitrary small positive number, form a discrete set.*†

THEOREM 11. *Let $f(x, y) \geqq 0$ be limited over the limited field $\mathfrak{A}$.*

*Bulletin of the American Mathematical Society, November, 1906, p. 61.

†Cf. SCHÖNFLIES: *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), p. 96.

Let $\underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}}$. *Then*

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

By theorem 3 we have

$$(1) \qquad \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

But $f - F \leqq 0$, hence by theorem 4 and *Lectures*, p. 535, we have

$$\underline{\int}_{\mathfrak{A}} f - F\bar{\mathfrak{A}} \leqq \underline{\int}_{\mathfrak{A}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f - F \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}.$$

And since by hypothesis $\bar{\mathfrak{A}} = \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}$, this gives

$$(2) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f.$$

Relation (A) then follows from (1) and (2).

THEOREM 12. Let $f(x, y) \leqq 0$ *be limited over the limited field* $\mathfrak{A}$. *Let* $\underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}}$. *Then*

$$(d) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \underline{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

THEOREM 13. *Let* $f(x, y)$ *be limited over the limited field* $\mathfrak{A}$. *Let* $\mathfrak{C}$ *be an integrable function of* x *and let* C *be any constant. Then*

$$(e) \qquad \bar{\int}_{\mathfrak{A}} (f + C) = \bar{\int}_{\mathfrak{A}} f + C\bar{\mathfrak{A}}.$$

$$(f) \qquad \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} (f + C) = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f + C \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}.$$

Relation (e) follows as an application of a corollary of RICHARDSON'S.*

By an application of relation (e) we get relation (f), thus

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} (f + C) = \bar{\int}_{\mathfrak{B}} \Big[\bar{\int}_{\mathfrak{C}} f + C\bar{\mathfrak{C}} \Big] \qquad \text{by } (e)$$

$$= \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f + C \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}$$

by hypothesis and Richardson's corollary.

* These Transactions, vol. 9 (1908), p. 350.

We may obtain similar relations for lower integrals.

THEOREM 14. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A}$ which is iterable relatively to x. Then*

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{C}} \bar{\int}_{\mathfrak{B}} f \leqq \bar{\int}_{\mathfrak{A}} f.$$

Since $f + F \geqq 0$, by theorem 3

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} (f + F) \leqq \bar{\int}_{\mathfrak{A}} (f + F).$$

By theorem 13 we have

$$\bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} (f + F) = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f + F \int_{\mathfrak{B}} \bar{\mathfrak{C}}$$

$$\bar{\int}_{\mathfrak{A}} (f + F) = \bar{\int}_{\mathfrak{A}} f + F\bar{\mathfrak{A}}.$$

Therefore

$$(1) \qquad \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{A}} f + F\left(\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}}\right).$$

Again, since $f - F \leqq 0$, by theorem 4

$$\underline{\int}_{\mathfrak{A}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F).$$

But by relations for lower integrals similar to those of theorem 13

$$\underline{\int}_{\mathfrak{A}} (f - F) = \underline{\int}_{\mathfrak{A}} f - F\bar{\mathfrak{A}}, \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F) = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f - F \int_{\mathfrak{C}} \bar{\mathfrak{C}}.$$

Therefore

$$(2) \qquad -F\left(\bar{\mathfrak{A}} - \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}\right) + \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f.$$

But $\mathfrak{A}$ being iterable, we have

$$(3) \qquad \bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}} = 0.$$

Therefore relation (A) follows at once from (1), (2) and (3).

Relation (A) may be written

$$(g) \qquad \underline{\int} f d\mathfrak{A} \leqq \underline{\int} dx \underline{\int} f dy \leqq \bar{\int} dx \bar{\int} f dy \leqq \bar{\int} f d\mathfrak{A}.$$

If the field $\mathfrak{A}$ is iterable relatively to y, we obtain in a similar way

$$(h) \qquad \underline{\int} f d\mathfrak{A} \leqq \underline{\int} dy \underline{\int} f dx \leqq \bar{\int} dy \bar{\int} f dx \leqq \bar{\int} f d\mathfrak{A}.$$

COROLLARY 1. *Let $f(x, y)$ be integrable over any limited field which is iterable relatively to both x and y. Let f also be integrable over any section of $\mathfrak{A}$ parallel to either axis. Then*

$$\int f d\mathfrak{A} = \int dx \int f dy = \int dy \int f dx.$$

This follows at once by putting

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \bar{\int} f d\mathfrak{A} = \int f d\mathfrak{A}$$

in relations (g) and (h) above. It gives *a very general condition for inversion* in iterated integrals.

In conclusion, it may be shown that *the equality*

$$\underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{G}} f = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} f$$

is not sufficient for the existence of

$$\int_{\mathfrak{B}} \int_{\mathfrak{G}} f.$$

For consider the following example. Let $\mathfrak{A} = \{x, y\}$, where $0 \leqq x = p/q \leqq 1$, $0 \leqq y \leqq 1/q$. For rational y let $f(x, y) = +1$ and for irrational y let $f(x, y) = -1$. Then $\mathfrak{A}$ is discrete and f is limited, hence

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{G}} f = \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{G}} f = 0$$

by theorem 14. But

$$\underline{\int}_{\mathfrak{G}} f = -\frac{1}{q}, \qquad \bar{\int}_{\mathfrak{G}} f = +\frac{1}{q}$$

for each x in $\mathfrak{B}$. Therefore

$$\int_{\mathfrak{G}} f$$

does not exist at any point in $\mathfrak{B}$ and

$$\int_{\mathfrak{B}} \int_{\mathfrak{G}} f$$

cannot then exist.

THE UNIVERSITY OF ILLINOIS.

ON A CLASS OF HYPERFUCHSIAN FUNCTIONS*

BY

CLYDE FIRMAN CRAIG

Introduction.

As a generalization of the fuchsian functions, PICARD introduced the hyperfuchsian. These he defined† as uniform functions of two independent complex variables which are absolutely invariant for an infinite discontinuous group of linear fractional transformations on the variables. Moreover the group, when expressed in its homogeneous form, is such as to leave an indefinite Hermitian form invariant.

In a series of memoirs‡ PICARD has established the existence and general theory of such functions. In a particular example he uses as variables the ratios of the values of the integral

$$\int [z(z-1)(z-x)(z-y)]^{-\frac{1}{2}} dz$$

taken along three independent contours in the associated Riemann surface, and generates the group by the motions of the two branch-points x and y. Following and extending PICARD's study of this particular group, ALEZAIS§ has considered the Riemann surface

$$w^3 = (z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$$

and the values of the moduli of periodicity of a set of associated linearly independent integrals of the first kind along a properly chosen system of cross-cuts on the surface. The group is generated by the monodromy of the four branch-points $\alpha, \beta, \gamma, \delta$, and contains PICARD's group as a subgroup. Considering the PICARD group to be of special interest, ALEZAIS studies it in some detail and

* Presented to the Society, December 31, 1908.

† Acta Mathematica, vol. 5 (1885), p. 121. See also R. ALEZAIS, *Sur une classe de fonctions hyperfuchsiennes*, etc., Paris (1901), p. 1. This thesis contains (p. 3) a complete bibliography of Picard's papers on the problem. An abridgment of the thesis under the same title is in the Annales de l'École Normale Supérieure, (3) vol. 19 (1902), p. 261.

‡ Acta Mathematica, vol. 2 (1882), p. 1, and vol. 5 (1885), p. 21; also several notes in Comptes Rendus, 1882–1885.

§ Loc. cit.

constructs quotients of the products of theta-functions of half-integer characteristics and zero arguments as hyperfuchsian functions belonging to the group.

The only other binomial Riemann surface of genus 3 having five branch-points may be written

$$w^4 = (z-a)(z-b)(z-c)^2(z-d)^2(z-e)^2.$$

The present paper determines the monodromy group of this surface and a set of hyperfuchsian functions belonging to its group. The fourth section contains an extension of a known result on the number of linearly independent moduli of an abelian integral of the first kind connected with a binomial Riemann surface. In the final section an upper limit is fixed for the number of generators of monodromy groups connected with binomial Riemann surfaces, and a method is given for picking out a set of transformations from which the generators may be determined.

§ 1. *The monodromy group.*

Consider the Riemann surface of genus 3,

$$w^4 = (z-a)(z-b)(z-c)^2(z-d)^2(z-e)^2, \tag{1}$$

where a, b, c, d, e may be assumed finite. We will denote the different values of w corresponding to the same value of z by w_1, w_2, w_3, w_4, and will assign the branch w_m to sheet m of the surface. The several values of w will be connected by the relation $w_m = i^{m-1} w_1$.

A set of linearly independent integrals of the first kind is selected as follows:

$$u_1 = \int \frac{dz}{w}; \qquad u_2 = \int (z-c)(z-d)(z-e)\frac{dz}{w^3};$$
$$u_3 = \int z(z-c)(z-d)(z-e)\frac{dz}{w^3}. \tag{2}$$

The branch-lines and cross-cuts for the surface are chosen as indicated in Fig. 1.

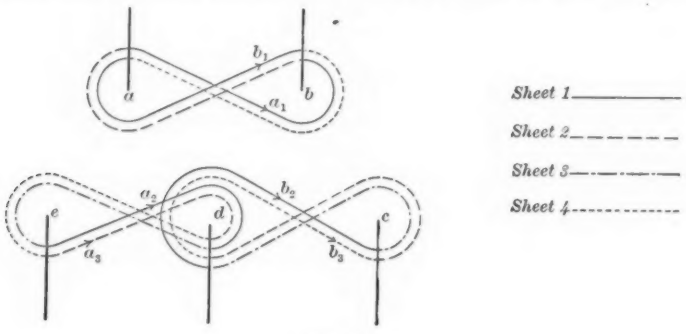

FIG. 1.

Denoting the results of integrating u_i along the paths a_j, b_j by $A_j^{(i)}$, $B_j^{(i)}$ respectively, the table of moduli of periodicity of the integrals (2) is

(3)

	a_1	a_2	a_3	b_1	b_2	b_3
u_1	$iA_1^{(1)}$	$-B_2^{(1)}$	$iB_2^{(1)}$	$A_1^{(1)}$	$A_2^{(1)}$	$-iA_2^{(1)}$
u_2	$-iA_1^{(2)}$	$-B_2^{(2)}$	$-iB_2^{(2)}$	$A_1^{(2)}$	$A_2^{(2)}$	$iA_2^{(2)}$
u_3	$-iA_1^{(3)}$	$-B_2^{(3)}$	$-iB_2^{(3)}$	$A_1^{(3)}$	$A_2^{(3)}$	$iA_2^{(3)}$

subject to the conditions

$$iA_1^{(1)}A_1^{(2)} - A_2^{(2)}B_2^{(1)} + A_2^{(1)}B_1^{(2)} = 0,$$

$$iA_1^{(1)}A_1^{(3)} - A_2^{(3)}B_2^{(1)} + A_2^{(1)}B_1^{(3)} = 0.$$

The normal table of moduli assumes the form

(3′)

	a_1	a_2	a_3	b_1	b_2	b_3
v_1	πi	0	0	$\pi i(i)$	$\pi i(-u)$	$\pi i(-iu)$
v_2	0	πi	0	$\pi i(-u)$	$\pi i\left(\frac{-iu^2}{2} - v\right)$	$\pi i\left(\frac{u^2}{2}\right)$
v_3	0	0	πi	$\pi i(-iu)$	$\pi i\left(\frac{u^2}{2}\right)$	$\pi i\left(\frac{iu^2}{2} - v\right)$

where

$$u = \frac{A_1^{(1)}}{B_2^{(1)}}, \qquad v = \frac{A_2^{(1)}}{B_2^{(1)}}.$$

Denote by a'_{ik} the real part of the element of this table which represents the modulus of periodicity of the integral v_k at the cut b_i. If r_i, r_k are integers, then * $\sum_{ik} a'_{ik} r_i r_k$ must be negative; this condition will be satisfied if †

$$u\bar{u} - i(v - \bar{v}) < 0, \tag{4}$$

wherein $\bar{u}$, $\bar{v}$, are the conjugates of u, v.

We now define the group G which we desire to investigate as the totality of linear fractional transformations on the variables u, v due to the monodromy of the branch-points of the given surface (1). The branch-points are supposed to move continuously over the plane in such a way that at the end of the movement they occupy as a whole their original positions and the Riemann surface has returned to its original form. Since $u = A_1^{(1)}/B_2^{(1)}$, $v = A_2^{(1)}/B_2^{(1)}$, and $A_1^{(1)}$, $A_2^{(1)}$, $B_2^{(1)}$ are the values of the integral u_1 along the paths a_1, a_2, b_2 respectively, in determining the transformation on u, v due to any movement of the

* STAHL, *Abelsche Funktionen*, p. 122 (XV).

† KRAZER, *Thetafunktionen*, p. 17 (XVII).

branch-points, it will be necessary merely to investigate the value of u_1 along the paths a_1', a_2', b_2', which will denote the paths into which a_1, a_2, b_2 have been changed. For brevity write

$$x = A_1^{(1)}, \qquad y = A_2^{(1)}, \qquad z = B_2^{(1)}. \tag{5}$$

The values of u_1 along the new paths will be denoted by x', y', z' respectively, and the variables u, v will have been transformed into the variables u', v', where

$$u' = \frac{x'}{z'}, \qquad v' = \frac{y'}{z'}.$$

By a positive interchange of two branch-points we shall mean that the positions are interchanged in such a manner that the line joining the two shall have been rotated counter-clockwise about its middle point. When necessary the actual paths of the points interchanged are supposed to vary from the circle described by the end points of the joining line sufficiently to prevent any third point being enclosed in the completed path of the two points. A point will be said to move around another point positively when the enclosed area is to the left of the moving point; this area will be assumed to enclose but a single branch-point unless the contrary is stated. Denote by $(\kappa\lambda)$ the transformation on the variables arising from a positive interchange of the two points κ and λ. This same notation will be used to denote the motion of κ positively around λ if κ and λ cannot be interchanged under the above restrictions on the movement of the branch-points.

By interchanging the branch-points a and b positively, it is readily found that x, y, z undergo the transformation

$$x' = ix, \qquad y' = y, \qquad z' = z;$$

whence

$$u' = -\frac{u}{i}, \qquad v' = \frac{iv}{i}, \tag{ab}$$

the coefficients being so written that their determinant is unity. Similarly we obtain

$$u' = u, \qquad v' = v - 1, \tag{cd}$$

$$u' = \frac{u}{v+1}, \qquad v' = \frac{v}{v+1}, \tag{de}$$

$$u' = \frac{-u+1+i}{i}, \qquad v' = \frac{(-1+i)u + iv + 1 - i}{i}. \tag{ae}$$

Every other monodromy of the branch-points which leaves the surface unaltered may be considered as a combination of the above four taken in proper

order, and consequently the transformation arising from such a monodromy may be expressed in terms of the above four. To show this we may proceed as follows, distinguishing two cases, namely, when a single pair of branch-points is moved, and when more than two are moved.

Under the first case we have operations such as interchanging c and e, or moving b around d. To produce the motion of b around d, for instance, we may perform in succession $(ab)^{-1}$, (de), (ae), $(de)^{-1}$, (ab); or (de), $(ab)^{-1}$, (ae), (ab), $(de)^{-1}$. Thus for our transformation on the variables we have

$$(bd) = (ab)^{-1}(de)(ae)(de)^{-1}(ab) = (de)(ab)^{-1}(ae)(ab)(de)^{-1},$$

where the transformations on the variables are applied from *right to left* as written. The last written equality may be easily verified; for (ab) and (de) are commutative, as are their powers, since the corresponding motions on the surface may be performed simultaneously without interference. Suppose now that one of the branch-points, say a, performs any circuit whatsoever and either returns to its original position or makes the possible interchange with the branch-point b. We may resolve its path (by proper deformation) into a series of circuits around the remaining branch-points singly, together with the possible interchange at the end. Since each of these circuits about the single branch-points and the interchange are severally expressible in terms of the four fundamental motions, it follows that the transformation due to the motion of a is expressible in terms of the four fundamental transformations. So with each of the other points b, c, d, e.

In case two or more branch-points move at the same time, no difficulty will arise if the paths do not intersect. For the paths are traversed independently and the movement giving rise to one path may be considered as having been completed before any other branch-point begins to move. The above argument shows that the transformation arising from these several motions can be expressed in terms of the four fundamental ones. If the paths of the moving points do intersect, we must specify the order in which two or more branch-points pass through a given point of intersection. For two branch-points must not coincide, and moreover the order specified must be compatible with the geometric possibilities of the surface. When this order is established the portions of the paths between the points of intersection may be considered as new paths of the form treated above. At the points of intersection we have simply an interchange of two points or one point moving around another. Since any such motion can now be resolved into the four fundamental ones, each taken the proper number of times and in the proper order, it follows that the transformation arising from any monodromy of the branch-points can be expressed in terms of (ab), (cd), (de), (ae).

These four fundamental transformations are expressible in terms of the two,

$$(6)\qquad A \equiv \begin{vmatrix} 1 & 1-i & 0 \\ 1-i & -1-i & -1 \\ 0 & 1 & 0 \end{vmatrix}, \qquad B \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix},$$

the determinant of the coefficients only being written. The expressions are

$$(ab) = BA^5B^{-1}A^5BA,$$
$$(cd) = B,$$
$$(de) = A^{-1}BA,$$
$$(ae) = A^{-2}B^{-1}A^{-2}B^{-1}.$$

Moreover A is a transformation of the given group, since

$$A = (ae)(cd)(de)(ab)^{-1}(cd).$$

Since B is non-periodic, the theorem follows:

The monodromy group G connected with the Riemann surface (1) is of infinite order and is generated by the two operators, A and B, defined by (6). The quadratic form $u\bar{u} - i(v - \bar{v})$ is a relative invariant for the group G.

§ 2. *The arithmetic form of the group.*

Since the coefficients of the generating transformations of G are complex integers, that is, are of the form $m + ni$, where m and n are integers, it follows that every transformation of G has complex integers for coefficients. We wish to characterize further these coefficients.

For this purpose write the general transformation of G in the form

$$(7)\quad u' = \frac{a_1u + b_1v + c_1}{\Delta}, \qquad v' = \frac{a_2u + b_2v + c_2}{\Delta}, \qquad \Delta = a_3u + b_3v + c_3,$$

where the coefficients are complex integers with determinant unity. Since the monodromy group leaves $\phi = i(v - \bar{v}) - u\bar{u}$ relatively invariant, the transformed form ϕ' will be connected with the original by the relation

$$\phi' = \frac{\phi}{\Delta\bar{\Delta}},$$

where $\bar{\Delta}$ is the conjugate of Δ. In order that this relation may subsist the coefficients in (7) must satisfy the conditions

$$(8)\quad \begin{aligned} &a_1\bar{a}_1 - i(a_2\bar{a}_3 - \bar{a}_2a_3) = 1, \qquad a_1\bar{b}_1 - i(a_2\bar{b}_3 - a_3\bar{b}_2) = 0, \\ &b_1\bar{b}_1 - i(b_2\bar{b}_3 - \bar{b}_2b_3) = 0, \qquad a_1\bar{c}_1 - i(a_2\bar{c}_3 - a_3\bar{c}_2) = 0, \\ &c_1\bar{c}_1 - i(c_2\bar{c}_3 - \bar{c}_3c_2) = 0, \qquad b_1\bar{c}_1 - i(b_2\bar{c}_3 - b_3\bar{c}_2) = -i. \end{aligned}$$

Using the relations (8) to solve (7), we obtain

$$\frac{u}{\Delta} = \bar{a}_1 u' - i\bar{a}_3 v' + i\bar{a}_2,$$

$$\frac{v}{\Delta} = i\bar{c}_1 u' + \bar{c}_3 v' - \bar{c}_2, \tag{7'}$$

$$\frac{1}{\Delta} = -i\bar{b}_1 u' - \bar{b}_3 u' + \bar{b}_2 = \Delta',$$

which give the inverse of transformation (7). On account of the relative invariance of ϕ', the coefficients of (7') must satisfy the relations

$$\begin{aligned} &a_1\bar{a}_1 + i(b_1\bar{c}_1 - \bar{b}_1 c_1) = 1, \qquad a_1\bar{a}_2 + i(b_1\bar{c}_2 - \bar{b}_2 c_1) = 0, \\ &a_2\bar{a}_2 + i(b_2\bar{c}_2 - \bar{b}_2 c_2) = 0, \qquad a_1\bar{a}_3 + i(b_1\bar{c}_3 - \bar{b}_3 c_1) = 0, \\ &a_3\bar{a}_3 + i(b_3\bar{c}_3 - \bar{b}_3 c_3) = 0, \qquad a_2\bar{a}_3 + i(b_2\bar{c}_3 - \bar{b}_3 c_2) = i. \end{aligned} \tag{8'}$$

Denote by $A_1, \cdots$ the minors of $a_1, \cdots$ in the determinant of (7); solve (7) for u/Δ, v/Δ, $1/\Delta$, and we obtain

$$\frac{u}{\Delta} = A_1 u' + A_2 v' + A_3,$$

$$\frac{v}{\Delta} = B_1 u' + B_2 v' + B_3,$$

$$\frac{1}{\Delta} = C_1 u' + C_2 v' + C_3.$$

Comparing this form of the inverse transformation with (7'), we obtain the relations

$$\begin{aligned} &\bar{a}_1 = b_2 c_3 - b_3 c_2 = A_1, && \bar{b}_1 = i(a_2 b_3 - a_3 b_2) = iC_1, \\ &\bar{a}_2 = -i(b_1 c_2 - b_2 c_1) = -iA_3, && \bar{b}_2 = a_1 b_2 - a_2 b_1 = C_3, \\ &\bar{a}_3 = -i(b_1 c_3 - b_3 c_1) = iA_2, && \bar{b}_3 = a_1 b_3 - a_3 b_1 = -C_2, \\ & && \bar{c}_1 = i(a_2 c_3 - a_3 c_2) = -iB_1, \\ & && \bar{c}_2 = a_1 c_2 - a_2 c_1 = -B_3, \\ & && \bar{c}_3 = a_1 c_3 - a_3 c_1 = B_2. \end{aligned} \tag{9}$$

These relations may be solved for a_1, a_2, a_3 (or a_1, b_1, c_1) in terms of the other elements, which will be connected by three bilinear relations taken from (8') [or (8)].

Since $(a_2\bar{a}_3 - \bar{a}_2 a_3)$ and similar expressions are divisible by 2, from (8) and (8′) it follows that

$$a_1\bar{a}_1 \equiv 1, \quad a_2\bar{a}_2 \equiv 0, \quad a_3\bar{a}_3 \equiv 0, \quad b_1\bar{b}_1 \equiv 0, \quad c_1\bar{c}_1 \equiv 0 \pmod 2,$$

from which

$$a_1 = 2m + 1 + 2ni \quad \text{or} \quad 2m + (2n+1)i,$$

$$a_2 \equiv 0, \quad a_3 \equiv 0, \quad b_1 \equiv 0, \quad c_1 \equiv 0 \ [\mathrm{mod}\,(1+i)].$$

From the relation $\bar{b}_2 = a_1 b_2 - a_2 b_1$ it follows that

$$\bar{b}_2 - a_1 b_2 \equiv 0 \pmod 2.$$

If $a_1 = 2m + 1 + 2ni$, this congruence gives $\bar{b}_2 - b_2 \equiv 0 \pmod 2$. If $a_1 = 2m + (2n+1)i$, the above congruence becomes

$$\bar{b}_2 - ib_2 \equiv 0 \pmod 2.$$

If we write $b_2 = b_2' + ib_2''$, this congruence becomes

$$(1-i)(b_2' + b_2'') \equiv 0 \pmod 2,$$

whence b_2' and b_2'' are of the same parity, and therefore $b_2 \equiv 0 \ [\mathrm{mod}\,(1+i)]$. A similar argument applied to the other relations (9) yields the conditions

$$b_3 \equiv 0, \quad c_2 \equiv 0, \quad c_3 \equiv 0 \ [\mathrm{mod}\,(1+i)].$$

Consequently $b_2c_3 - b_3c_2 \equiv 0 \pmod 2$. But $\bar{a}_1 = b_2c_3 - b_3c_2 \equiv 0 \pmod 2$, which violates the assumption

$$a_1 = 2m + (2n+1)i,$$

and hence this value of a_1 is impossible. We accordingly have

$$a_1 \equiv 1 \pmod 2.$$

Therefore $a_1 = 1 + 2m + 2ni$, and by (8) and (9),

$$b_2c_3 - b_3c_2 \equiv 1, \quad b_2\bar{c}_3 - \bar{b}_3c_2 \equiv 1 \pmod 2.$$

The residues of complex integers (mod 2) are $0, 1, i, 1+i$. The two above congruences reduce the possible combinations of residues of b_2, b_3, c_2, c_3 to 48. Using these 48 possible residues in connection with the equations (9), we are led to 192 transformations which are incongruent modulo 2.

By using the relations (9) the periods of the transformations of G may be readily determined. The characteristic equation for the general transformation of the group is

$$\begin{vmatrix} a_1 - \rho & b_1 & c_1 \\ a_2 & b_2 - \rho & c_2 \\ a_3 & b_3 & c_3 - \rho \end{vmatrix} = 0,$$

or, by (9),

$$\rho^3 - d\rho^2 + d\rho - 1 = 0, \qquad d = a_1 + b_2 + c_3 = a + bi,$$

a and b being integers. If the transformation is periodic, all the roots of the characteristic equation must be roots of unity, and conversely. Hence $|d| \leqq 3$. Let $\rho = (1 + it)/(1 - it)$, and the characteristic equation becomes

$$(1 + a)t^3 + bt^2 + (a - 3)t + b = 0.$$

If the roots of this equation are real, those of the characteristic equation are roots of unity, and conversely. The roots of this equation will be real if

$$G^2 + 4H^3 \leqq 0,$$

where

$$G = \tfrac{2}{27}(9a^2b + 36ab + 27b + b^3),$$

$$H = \frac{3(a + 1)(a - 3) - b^2}{9}.$$

The possible values of d for which this equation is satisfied are $-1, 0, 1, -1 \pm 2i, 2, \pm i, -1 \pm i, 3$. Construct the corresponding equation in ρ and factor; the periods follow immediately. With the notation

$$\varepsilon = e^{\frac{\pi i}{3}}, \qquad i = e^{\frac{\pi i}{2}}, \qquad j = e^{\frac{\pi i}{4}}$$

the tabulated results are:

d	-1	0	1	$-1 \pm 2i$	2	$\pm i$	$-1 \pm i$	3
ρ	$1, -1, -1$	$1, -\varepsilon, \varepsilon^2$	$1, i, -i$	$-1, \pm i, \pm i$	$1, \varepsilon, -\varepsilon^2$	$i, j, -j, -i, ij, -ij$	$-1, \pm i\varepsilon, \mp i\varepsilon^2,$	$1, 1, 1$
Period	$2, \infty$	3	4	$4, \infty$	6	8	12	∞

For $d = -1, -1 \pm 2i$ all second order minors of the determinant of the characteristic equation must vanish for the double root if the period is finite. For all values of d other than the above the substitution is non-periodic.*

* The group G, considered as abstract, is apparently characterized by only the conditions of this section. The problem arises: Is G the maximum group which satisfies these conditions? The question is as yet unanswered; but there is good reason to believe that G is the maximum group. If we transform G into a group G' by means of

$$\begin{vmatrix} -i & 1 & 1 \\ -1 & 0 & -i \\ 0 & 1 & 1 \end{vmatrix},$$

then the coefficients of each linear transformation in G' will be complex integers with determinant 1. Moreover the invariant locus for the new group is the hypersphere $u\bar{u} + v\bar{v} = 1$. The group G' is apparently the maximum group of substitutions having these restrictions, and would accordingly be the simplest analogue, in the case of two variables, to the modular group of linear transformations on a single complex variable.

§ 3. *The hyperfuchsian functions.*

Using the general transformation of the group G in its homogeneous form

$$(7'') \qquad \begin{aligned} x' &= a_1 x + b_1 y + c_1 z, \\ y' &= a_2 x + b_2 y + c_2 z, \\ z' &= a_3 x + b_3 y + c_3 z, \end{aligned}$$

we see that corresponding to this linear transformation on the variables, the elements of the table of moduli (3) undergo a linear transformation. The corresponding transformation on the allied theta series requires the determination of a set of integral coefficients $\alpha_{\iota\kappa}$, $\beta_{\iota\kappa}$, $\gamma_{\iota\kappa}$, $\delta_{\iota\kappa}$ which satisfy the identities

$$-B_\kappa^{(i)} = \sum_{\mu=1}^{3} (-\alpha_{\kappa\mu} B_\mu^{(i)} + \beta_{\kappa\mu} A_\mu^{(i)}),$$

$$A_\kappa'^{(i)} = \sum_{=}^{3} (-\gamma_{\kappa\mu} B_\mu^{(i)} + \delta_{\kappa\mu} A_\mu^{(i)}),$$

wherein $A_\kappa^{(i)}$, etc., have the meanings assigned in § 1, and $A_\kappa'^{(i)}$ is the value of $A_\kappa^{(i)}$ after transformation. Replacing $A_\kappa^{(i)}$, $A_\kappa'^{(i)}$, etc., in the above identities by their values given by (5) and (7''), the theta transformation is given by

$$\left|\begin{array}{ccc|ccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ -\beta_{21} & -\alpha_{23} & \alpha_{22} & \alpha_{21} & -\beta_{23} & \beta_{22} \\ \hline -\beta_{11} & -\alpha_{13} & \alpha_{12} & \alpha_{11} & -\beta_{13} & \beta_{12} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \delta_{21} & \delta_{22} & \delta_{23} \\ -\delta_{21} & -\gamma_{23} & \gamma_{22} & \gamma_{21} & -\delta_{23} & \delta_{22} \end{array}\right|$$

wherein the coefficients are subject to certain well-known bilinear relations, and are connected with the coefficients of (7'') by means of the relations

$$(10) \qquad \begin{aligned} a_1 &= \alpha_{11} - \beta i_{11}, & b_1 &= -\beta_{13} - i\beta_{12}, & c_1 &= \alpha_{13} + i\alpha_{12}, \\ a_2 &= \delta_{21} + i\gamma_{21}, & b_2 &= \delta_{22} - i\delta_{23}, & c_2 &= -\gamma_{22} + i\gamma_{23}, \\ a_3 &= -\beta_{21} - i\alpha_{21}, & b_3 &= -\beta_{22} - i\beta_{23}, & c_3 &= \alpha_{22} - i\alpha_{23}. \end{aligned}$$

The bilinear relations mentioned above, when taken by columns, reduce to the following:

$$\alpha_{11}^2 + 2\alpha_{21}\delta_{21} + \beta_{11}^2 - 2\beta_{21}\gamma_{21} = 1,$$

$$-\alpha_{12}\beta_{13} + \alpha_{13}\beta_{12} + \alpha_{22}\delta_{22} + \alpha_{23}\delta_{23} - \beta_{22}\gamma_{22} - \beta_{23}\gamma_{23} = 1,$$

$$\alpha_{11}\alpha_{13} - \alpha_{12}\beta_{11} - \alpha_{21}\gamma_{22} + \alpha_{22}\gamma_{21} + \alpha_{23}\delta_{21} - \beta_{21}\gamma_{23} = 0,$$

$$\alpha_{11}\alpha_{12} + \alpha_{13}\beta_{11} + \alpha_{21}\gamma_{23} + \alpha_{22}\delta_{21} - \alpha_{23}\gamma_{21} - \beta_{21}\gamma_{22} = 0,$$

$$\alpha_{11}\beta_{13} - \alpha_{21}\delta_{22} - \beta_{11}\beta_{12} - \beta_{21}\delta_{23} + \beta_{22}\gamma_{21} + \beta_{23}\delta_{21} = 0,$$

$$\alpha_{11}\beta_{12} + \alpha_{21}\delta_{23} + \beta_{11}\beta_{13} - \beta_{21}\delta_{22} + \beta_{22}\delta_{21} - \beta_{23}\gamma_{21} = 0,$$

$$\alpha_{12}\beta_{12} + \alpha_{13}\beta_{13} + \alpha_{22}\delta_{23} - \alpha_{23}\delta_{22} + \beta_{22}\gamma_{23} - \beta_{23}\gamma_{22} = 0,$$

$$\alpha_{12}^2 + \alpha_{13}^2 + 2\alpha_{22}\gamma_{23} - 2\alpha_{23}\gamma_{22} = 0,$$

$$\beta_{12}^2 + \beta_{13}^2 + 2\beta_{22}\delta_{23} - 2\beta_{23}\delta_{22} = 0.$$

Rewrite the first, ninth, and eighth of these equations as indicated below. Multiply the sixth equation by i, and subtract the fifth from the result. Multiply the fourth by $-i$, and add the third to the result. From the second multiplied by $-i$ subtract the seventh. The results of these operations in the order indicated are

$$(\alpha_{11} - i\beta_{11})(\alpha_{11} + i\beta_{11}) - i(\delta_{21} + i\gamma_{21})(-\beta_{21} + i\alpha_{21}) + i(\delta_{21} - i\gamma_{21})(-\beta_{21} - i\alpha_{21}) = 1,$$

$$(-\beta_{13} - i\beta_{12})(-\beta_{13} + i\beta_{12}) - i(\delta_{22} - i\delta_{23})(-\beta_{22} - i\beta_{23}) + i(\delta_{22} + i\delta_{23})(-\beta_{22} + i\beta_{23}) = 0,$$

$$(\alpha_{13} + i\alpha_{12})(\alpha_{13} - i\alpha_{12}) - i(-\gamma_{22} + i\gamma_{23})(\alpha_{22} + i\alpha_{23}) + i(-\gamma_{22} - i\gamma_{23})(\alpha_{22} - i\alpha_{23}) = 0,$$

$$(\alpha_{11} - i\beta_{11})(-\beta_{13} + i\beta_{12}) - i(\delta_{21} + i\gamma_{21})(-\beta_{22} - i\beta_{23}) + i(-\beta_{21} - i\alpha_{21})(\delta_{22} + i\delta_{23}) = 0,$$

$$(\alpha_{11} - i\beta_{11})(\alpha_{13} - i\alpha_{12}) - i(\delta_{21} + i\gamma_{21})(\alpha_{22} + i\alpha_{23}) + i(-\beta_{21} - i\alpha_{21})(-\gamma_{22} - i\gamma_{23}) = 0,$$

$$(-\beta_{13} - i\beta_{12})(\alpha_{13} - i\alpha_{12}) - i(\delta_{22} - i\delta_{23})(\alpha_{22} + i\alpha_{23}) + i(-\beta_{22} + i\beta_{23})(-\gamma_{22} - i\gamma_{23}) = -i.$$

In these equations substitute the values of a_i, b_i, c_i from (10), and we obtain the fundamental equations of condition (8). In like manner the bilinear relations obtained by taking the elements of the theta-transformation by rows may be combined to produce a set of equations similar to the above, out of which the conditions of (8′) can be formed.

Since the equations (9) are consequences of the conditions (8) and (8′), we may substitute in (9) the values of a_i, b_i, c_i given by (10). Separating the real and imaginary parts of the resulting equations, we obtain a solution of the bilinear relations connecting the coefficients of the theta transformation. This solution gives six coefficients explicitly in terms of the remaining twelve, which are connected by four bilinear relations.

Because of the particular moduli of the problem at hand certain linear relations exist among the theta functions. To determine what these are we follow the values of any theta function under the transformation which corresponds to identity in G. For this latter $a_1 = b_2 = c_3$, $a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$.

Using (10), we find the theta transformation to be

$$\left(\begin{array}{ccc|ccc} m & 0 & 0 & n & 0 & 0 \\ 0 & m & n & 0 & 0 & 0 \\ 0 & -n & m & 0 & 0 & 0 \\ \hline -n & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & -n & m \end{array}\right)$$

with $m^2 + n^2 = 1$, and m, n integers. This transformation yields four cases, viz.: $m = 1$, $n = 0$; $m = -1$, $n = 0$; $m = 0$, $n = +1$; $m = 0$, $n = -1$. The corresponding relations between the theta functions with zero arguments are *

$$\begin{aligned} &\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) = \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)), \\ &\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) = \vartheta\begin{bmatrix} -g \\ -h \end{bmatrix}((0)), \\ &\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) = e^{2\pi i g_1 h_1}\vartheta\begin{bmatrix} -h_1 & g_3 & -g_2 \\ g_1 & h_3 & -h_2 \end{bmatrix}((0)), \\ &\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) = e^{2\pi i g_1 h_1}\vartheta\begin{bmatrix} h_1 & -g_3 & g_2 \\ -g_1 & -h_3 & h_2 \end{bmatrix}((0)). \end{aligned} \tag{11}$$

By using the relations (10) the theta-transformations corresponding to the generators of G are determined to be

$$A: \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \end{array}\right), \qquad B^n: \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & n & 0 & 0 & 1 & 0 \\ 0 & 0 & n & 0 & 0 & 1 \end{array}\right),$$

and the corresponding transformations on the theta-functions with zero arguments are †:

$$\begin{aligned} A:\ &\vartheta\begin{bmatrix} g_1-h_2+h_3 & h_2 \\ h_1+h_2+h_3 & g_1-g_2+h_1-h_2+h_3 \end{bmatrix}\begin{matrix} g_1-g_3-h_1-h_2-h_3 \end{matrix}((0))_b \\ &\qquad = ive^{-2\pi i[h_2(h_2-g_1+g_2)+h_3(g_1-g_3)]}\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0))_a, \\ B^n:\ &\vartheta\begin{bmatrix} g_1 & g_2 & g_3 \\ h_1 & \frac{1}{2}n-h_2g_2+h_2 & \frac{1}{2}n-h_2g_3+h_3 \end{bmatrix}((0))_b = e^{n\pi i[g_2+g_3-g_2^2-g_3^2]}\vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0))_a. \end{aligned} \tag{12}$$

* KRAZER u. PRYM, *Neue Grundlagen einer Theorie der allgemeinen Thetafunktionen*, etc., p. 121.

† Loc. cit.

We now wish to determine the simplest possible products of theta functions with zero arguments which are relatively invariant under the group G. If a product of two functions ϑ_1 and ϑ_2 is relatively invariant, then it follows that ϑ_1 is a relative invariant under A^2 and B^2. For this to be true, the transformed characteristic must be reducible to one of the four given in (11). In addition to this, the characteristic of ϑ_1 must be transformed into that of ϑ_2 by A or B. This requires further conditions similar to the above. The characteristic of ϑ_1 will then be the simultaneous solution of these sets of conditions, if such a solution exists. A similar procedure may be used to determine whether the product of several theta-functions can be relatively invariant under G or not. The following products are believed to be the simplest:

$$
\begin{aligned}
\phi_1 &\equiv \begin{bmatrix} 000 \\ 000 \end{bmatrix}\begin{bmatrix} 000 \\ 011 \end{bmatrix}\begin{bmatrix} 011 \\ 000 \end{bmatrix}, \\
\phi_2 &\equiv \begin{bmatrix} 000 \\ 100 \end{bmatrix}\begin{bmatrix} 000 \\ 111 \end{bmatrix}\begin{bmatrix} 011 \\ 100 \end{bmatrix}\begin{bmatrix} 011 \\ 111 \end{bmatrix}, \\
\phi_3 &\equiv \begin{bmatrix} 000 \\ 110 \end{bmatrix}\begin{bmatrix} 000 \\ 101 \end{bmatrix}\begin{bmatrix} 001 \\ 100 \end{bmatrix}\begin{bmatrix} 001 \\ 110 \end{bmatrix}\begin{bmatrix} 010 \\ 100 \end{bmatrix}\begin{bmatrix} 010 \\ 101 \end{bmatrix}, \\
\phi_4 &\equiv \begin{bmatrix} 000 \\ 010 \end{bmatrix}\begin{bmatrix} 010 \\ 000 \end{bmatrix}\begin{bmatrix} 010 \\ 001 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix}\begin{bmatrix} 101 \\ 111 \end{bmatrix}\begin{bmatrix} 111 \\ 101 \end{bmatrix},
\end{aligned}
\tag{13}
$$

in which for the sake of brevity we write

$$
\begin{bmatrix} \lambda \\ \mu \end{bmatrix} \equiv \vartheta \begin{bmatrix} \frac{1}{2}\lambda \\ \frac{1}{2}\mu \end{bmatrix}((0))_a
$$

(λ, μ integers). The function $\phi \equiv \left[\begin{smallmatrix} 011 \\ 011 \end{smallmatrix}\right]$ is also relatively invariant, but from (11) we find that $\phi = -\phi$, and hence that $\phi \equiv 0$. Denote by ϕ_i' the transformed form of ϕ_i. By using (11) and (12) the following table is deduced for the generating transformations:

	ϕ_1'	ϕ_2'	ϕ_3'	ϕ_4'
A	$-v^3\phi_1$	$-v^4\phi_2$	$v^6\phi_3$	$v^6\phi_4$
B	$i\phi_1$	$-\phi_2$	$-\phi_3$	$-\phi_4$

None of the functions ϕ_i vanish identically. To show this it will be sufficient to find values of u, v for which some one theta function of each ϕ_i is different from zero, since the factors of each product are permuted by the transformations of the group. Take the particular values $u = 0$, $v = -i$, which satisfy the condition (4), and substitute in the functions $\left[\begin{smallmatrix} 000 \\ 000 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 000 \\ 100 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 100 \\ 000 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 010 \\ 100 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 101 \\ 000 \end{smallmatrix}\right]$,

and $\left[\begin{smallmatrix}010\\000\end{smallmatrix}\right]$. This gives

$$\left[\begin{matrix}000\\000\end{matrix}\right] = \sum_{m_i=-\infty}^{\infty} e^{-\pi[m_1^2+m_2^2+m_3^2]},$$

$$\left[\begin{matrix}100\\000\end{matrix}\right] = \sum_{m_i=-\infty}^{\infty} e^{-\pi[(m_1+\frac{1}{2})^2+m_2^2+m_3^2]},$$

$$\left[\begin{matrix}101\\000\end{matrix}\right] = \sum_{m_i=-\infty}^{\infty} e^{-\pi[(m_1+\frac{1}{2})^2+m_2^2+(m_3+\frac{1}{2})^2]},$$

$$\left[\begin{matrix}010\\000\end{matrix}\right] = \sum_{m_i=-\infty}^{\infty} e^{-\pi[m_1^2+(m_2+\frac{1}{2})^2+m_3^2]},$$

which evidently do not vanish.

None of the functions ϕ_1, ϕ_2, ϕ_3 vanishes identically, and the functions

$$F_1 = \frac{\phi_3}{\phi_1^2}, \qquad F_2 = \frac{\phi_4}{\phi_1^2} \tag{15}$$

are absolutely invariant under the group G. These functions are believed to be as simple as any of the kind under discussion and belonging to the group.*

§ 4. *Linear independence of the moduli.*

It will be observed that in Table (3) the six moduli of a given integral u_i are expressed linearly in terms of three such. This is a particular case of a general theorem which we proceed to deduce.

Denote by

$$u_i = \int \frac{R_i(z)\,dz}{w^{\eta_i}} \qquad (i = 1, 2, \cdots, p),$$

where $R_i(z)$ is a rational integral function of z, and η_i is an integer, a system of p linearly independent integrals of the first kind connected with the Riemann surface

$$w^\nu = (z-a)^\alpha (z-b)^\beta \cdots (z-n)^\lambda$$

having μ branch-points which may be assumed finite, and with

$$0 < \alpha < \nu,\ 0 < \beta < \nu,\ \cdots,\ 0 < \lambda < \nu,\ \alpha + \beta + \cdots + \lambda \equiv 0 \pmod{\nu}.$$

Denote the different values of w corresponding to the same value of z by $w_1, w_2, \cdots, w_\nu$, and assign the branch w_m to the mth sheet of the surface.

*PICARD has shown (Acta Mathematica, vol. 5 (1884), p. 175) that we can choose two hyperfuchsian functions, relative to a group G, such that every other function belonging to the same group is an algebraic function of the two chosen. Whether or not the two functions (15) are suitable for that purpose cannot be determined without a knowledge of the fundamental region for G.

The several values of w will be connected by the relation

$$w_m = \theta^{(m-1)} w_1, \qquad \theta = e^{\frac{2\pi i}{\nu}}.$$

Denote by u any integral of the first kind, and by $u^{(m)}$ the value of u taken over any path starting from a point in sheet m. Then

$$(17) \qquad u^{(m)} = \int \frac{R(z)\,dz}{w_m^\eta} = \int \frac{R(z)\,dz}{\theta^{(m-1)\eta} w_1^\eta} = \theta^{\rho_m} \int \frac{R(z)\,dz}{w_1^\eta} = \theta^{\rho_m} u^{(1)},$$

where $(m-1)\eta + \rho_m \equiv 0 \pmod{\nu}$, with $0 \leqq \rho_m < \nu$.

By a simple circuit about a branch-point is meant a loop which starts and ends at z_0 in two different sheets which are successive in the order of the winding of the sheets for that branch-point. Denote by $\alpha^{(m)}$, $\beta^{(m)}$, $\cdots$, the values of $u^{(m)}$ along simple circuits about the branch-points a, b, $\cdots$, respectively. The values of $\alpha^{(m)}$, $\beta^{(m)}$, $\cdots$, for the different values of m are connected by the relation (17) as follows: $\alpha^{(m)} = \theta^{\sigma_m}\alpha^{(1)}$, $\beta^{(m)} = \theta^{\sigma'_m}\beta^{(1)}$, $\cdots$, where σ_m, σ'_m, etc., are defined similarly to ρ_m. Since any closed path on the surface can be resolved by continuous deformation into a set of simple circuits about the branch-points, it follows that the value of $u^{(m)}$ along any closed path may be expressed in the form

$$c'_1\alpha^{(1)} + c'_2\beta^{(1)} + \cdots + c'_\mu \lambda^{(1)},$$

where c'_e is a sum of roots of unity. For the integrals along the several simple circuits about a may each be expressed in the form $\pm\theta^{\sigma_m}\alpha^{(1)}$, depending on the direction in which the integral is taken; consequently the portion of the value of $u^{(m)}$ due to paths around a is of the form $c'_1\alpha^{(1)}$ where c'_1 is a sum of roots of unity. Similarly for the other branch-points. In particular, take u positively over a path around all the branch-points, and we obtain a relation of the form

$$(18) \qquad c_1\alpha^{(1)} + c_2\beta^{(1)} + \cdots + c_\mu\lambda^{(1)} = 0,$$

since the given path may be deformed into a point path.

Assume an appropriate set of branch-lines and cross-cuts, and define A_j, B_j, as in § 1. Since the moduli of periodicity of u are the results of integrating u along these cross-cuts of the surface, we have as the values of A_j, B_j

$$(19) \qquad \begin{aligned} A_j &= c_{1j}\alpha^{(1)} + c_{2j}\beta^{(1)} + \cdots + c_{\mu j}\lambda^{(1)}, \\ B_j &= d_{1j}\alpha^{(1)} + d_{2j}\beta^{(1)} + \cdots + d_{\mu j}\lambda^{(1)}, \end{aligned} \qquad (j = 1, 2, \cdots, p),$$

where c_{ij}, d_{ij} are constants. Substitute the value of $\lambda^{(1)}$ from (18) in the equations (19) and we have $2p$ equations expressing the moduli of periodicity A_j, B_j in terms of $\mu - 1$ quantities $\alpha^{(1)}$, $\beta^{(1)}$, $\cdots$. Not more than $\mu - 2$ of the $2p$ moduli A_j, B_j $(j = 1, 2, \cdots p)$ are linearly independent. To prove this, assume that $\mu - 1$ of them are independent. Solve these $\mu - 1$ linearly independent

equations for each of the $\mu - 1$ quantities $\alpha^{(1)}, \beta^{(1)}, \cdots$. Thus we find that each of the quantities $\alpha^{(1)}, \beta^{(1)}, \cdots$ is expressed as a linear function of $\mu - 1$ constants A_j, B_j. In particular, then,

$$\alpha^{(1)} = \text{const.}$$

But since $\alpha^{(1)}$ is the value of $w^{(1)}$ along an open path on the Riemann surface, it is a function of the end points and is not constant. Consequently the assumption made is false, and we have the theorem: *Not more than $\mu - 2$ of the moduli of periodicity of an abelian integral of the first kind associated with a binomial Riemann surface having μ branch-points are linearly independent.* *

§ 5. *The fundamental transformations for the group of the general binomial surface.*

In § 1 certain fundamental transformations arising from the monodromy of the branch-points of a Riemann surface were determined, and from them the generating tranformations of the group were found. The method employed there will be generalized in the present section; by analyzing separate cases we are finally led to the general theorem. We assume the conventions and definitions of § 1.

Consider the binomial Riemann surface

$$w^\nu = (z - a_1)^\alpha (z - a_2)^\alpha \cdots (z - a_m)^\alpha,$$

where $m\alpha \equiv 0 \pmod{\nu}$, $0 < \alpha < \nu$, α and ν integers and relative primes. Let z_0 be any arbitrarily chosen point of the surface, and with it as center swing a radius vector counter-clockwise; let the branch-points be numbered $1, 2, \cdots, m$, in the order in which the radius vector meets them. Denote by $(kl) = (lk)$ the transformation produced on any set of variables, functionally connected with the branch points a_k, a_l of the surface, due to the interchange of these branch-points. In the following discussion the symbol stands for the interchange itself. The interchange of the points is to be made in the manner indicated on pp. 40, 41.

If a_i, a_k, a_l are three branch-points such that $i < k < l$, then (il) is equivalent to either $t_1 = (ik)^{-1}(kl)(ik)$ or $t_2 = (kl)(ik)(kl)^{-1}$. For in t_1, $(ik)^{-1}$ means the interchange of a_i and a_k clockwise, then (kl) the interchange of a_l and a_k ($\equiv a_i$) counter-clockwise, thus putting a_i in the place of a_l, then (ik) the interchange of a_k ($\equiv a_l$) and a_i ($\equiv a_k$) putting a_l in place of a_i and the path of a_k may be deformed into a point path without passing over a branch-point: the paths of a_i and a_k are between z_0 and a_l. A similar analysis verifies t_2. An analogous procedure applies if four or more branch-points are permuted to obtain an interchange. In particular, if $k < l$, we may express (kl) in terms

* See FRICKE-KLEIN, *Automorphe Functionen*, II, p. 135, where the essential statement of this theorem for $\mu = 4$ is credited to J. WELLSTEIN.

of successive interchanges of branch-points, and obtain

$$(kl) = (k, k+1)^{-1}(k+1, k+2)^{-1} \cdots (l-2, l-1)^{-1}(l-1, l)(l-1, l-2)\cdots(k+1, k).$$

It follows that the transformations due to every simple interchange can be expressed in terms of $m-1$ transformations $(i, i+1)$, $i = 1, 2, \cdots, m-1$.

We will next discuss all other admissible types of motions of the branch-points and the corresponding transformations of the variables. Let a_k perform a complete loop about a_l; the resultant transformation is $(kl)^2$, for the complete loop is the succession of two interchanges of the points a_k and a_l. In such a case the path of neither point can be deformed to a point-path. Next assume that a_k and a_l $(k < l)$ are interchanged so that a branch-point a_i is enclosed in the total path of the other points. The resultant transformation is

$$(ki)(il)(ki) = (il)(ki)(il),$$

for in either case the path of a_i is a point-path, that of a_l is between z_0 and a_i while a_i is between z_0 and that of a_k. If a_j is situated similarly to a_i then the resulting transformation is

$$(ki)(ij)(jl)(ij)(ik) = (lj)(ji)(ik)(ji)(jl).$$

In case a_i lies between z_0 and the paths of a_k, a_l, the corresponding transformation is

$$(ki)(il)(ik)^{-1} = (il)^{-1}(ik)(il),$$

and similarly for more points situated like a_i.

By an argument similar to that on p. 41 we may resolve any motion of any number of branch-points into a sequence of motions of the types discussed above, and may consequently express the resultant transformations in terms of $m-1$ successive interchanges designated above.

Next consider the binomial Riemann surface

$$w^\nu = (z-a_1)^\alpha \cdots (z-a_m)^\alpha (z-b_1)^\beta \cdots (z-b_n)^\beta,$$

where $0 < \alpha \neq \beta < \nu$, $m\alpha + n\beta \equiv 0 \pmod{\nu}$, α, β, and ν are integers, and either α or β is relatively prime to ν. In a manner similar to that above we may select two sets of $m-1$, $n-1$ transformations in terms of which all the transformations due to all motions of the points a among themselves, and to the points b among themselves may be expressed. Denote by T_{12} the transformation arising from the motion of some point a around some point b once. Evidently the points can be so chosen that the last point of the set a moves around the first point of the set b. Then every transformation arising from proper motions of the points a, b can be expressed in terms of the $m+n-1$ indicated transformations. To prove this it will evidently be sufficient to express the

transformation arising from the motion of an arbitrary point a around an arbitrary point b. Let a_i and b_j be the points. Interchange a_j and a_m, b_j and b_1; then use T_{12}, which puts a_m around b_1; then interchange a_i and a_m, b_j and b_1, care being taken as to the direction of interchange. This completes the motions, and all paths except those of a_i, b_j are point-paths, thus furnishing the required representation of the transformation. In general the number $m+n-1$ of fundamental transformations cannot be reduced. If, however, $n=1$, then $T_{12}=(m, m-1)^{-1}\cdots(12)^{-2}\cdots(m, m-1)^{-1}$, and the number of fundamental transformations is not greater than $m-1$.

In the case of several sets of branch-points $m_1, m_2, \cdots, m_n$ in number, which cannot be interchanged, a fundamental transformation will be required corresponding to the motion of some point of a set around a point of every other set. The corresponding transformations will be designated by T_{kl}, where k and l are different and refer to the above n sets of branch-points. The number of these will be $\frac{1}{2}n(n-1)$, and this number will diminish by unity if some m_i is unity. The number of transformations of each set is known from the above discussion to be m_i-1. To sum up, we have the following theorem:

The number of generators of the group arising from all possible motions of the branch-points of the binomial Riemann surface

$$w^\nu=\prod_{m_1}(z-a_i)^{\alpha_1}\prod_{m_2}(z-b_i)^{\alpha_2}\cdots\prod_{m_n}(z-l_i)^{\alpha_n},$$

where

$$0<\alpha_i\neq\alpha_j<\nu, \qquad \sum_{i=1}^{n}m_i\alpha_i\equiv 0 \pmod{\nu},$$

α_i and ν are integers, and some α_i is relatively prime to ν, does not exceed

$$\sum_{i=1}^{n}m_i-n+\frac{n(n-1)}{2}=\sum_{i=1}^{n}m_i+\frac{n(n-3)}{2}.$$

If a detailed knowledge of an individual case is available, this number may be reduced. The group of § 1 is an instance in point.

CORNELL UNIVERSITY,
Ithaca, N. Y.

PERIODIC ORBITS ABOUT AN OBLATE SPHEROID

BY

WILLIAM DUNCAN MACMILLAN

§ 1. *Introduction.*

The orbit of a particle about an oblate spheroid is not in general closed geometrically. The motion of the particle is not, therefore, in general, periodic from a geometric point of view. But if we consider the orbits as described by the particle in a revolving meridian plane which passes constantly through the particle several classes of closed orbits can be found in which the motion is periodic. The failure of these orbits to close in space arises from the incommensurability of the period of rotation of the line of nodes with the period of motion in the revolving plane. When these periods happen to be commensurable the orbits are closed in space and the motion is therefore periodic, though the period may be very great. Indeed, it seems that most of the difficulty in giving mathematical expressions for the orbits about an oblate spheroid rests upon the question of incommensurability of periods. The difficulty arising from the motion of the node can be overcome by the use of the revolving plane, but other incommensurabilities, such as that introduced by the eccentricity, can not be eliminated in this manner.

Orbits closed in the revolving plane are considered most conveniently in two general classes: (1) Those which reënter after one revolution, (2) those which reënter after many revolutions. The existence of both classes is established in this paper and convenient methods for constructing the solutions are given. Orbits which reënter after the first revolution are naturally the simpler and will be considered in the first part of the paper. Those lying in the equatorial plane of the spheroid become straight lines in the revolving plane, and it is shown that within the realm of convergence of the series employed all orbits in the equatorial plane are periodic. When the orbits do not lie in the equatorial plane there exists one, and only one, orbit for any arbitrarily assigned values of the inclination and the mean distance. These orbits reduce to circles with the vanishing of the oblateness of the spheroid.

In considering orbits which reënter only after many revolutions the differential equations are found to be very complex, and one would despair of proving the existence of these orbits by the ordinary direct computation of the necessary coefficients. However, a proof of their existence and a method for the construc-

tion of the solutions are given by the aid of certain theorems, which are here established, on the character of the solutions of non-homogeneous linear differential equations with periodic coefficients.

These periodic orbits of many revolutions involve five constants, four of which are entirely arbitrary, and the fifth subject only to the limitation that it shall satisfy certain commensurability conditions. One constant, only, is missing for a complete integration of the differential equations. These orbits are all symmetric with respect to the equatorial plane.

§ 2. *The differential equations.*

The differential equations of motion of a particle about an oblate spheroid are*

$$(1)\qquad \begin{aligned} \frac{d^2x}{dt^2} &= -\frac{k^2Mx}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial x},\\ \frac{d^2y}{dt^2} &= -\frac{k^2My}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial y},\\ \frac{d^2z}{dt^2} &= -\frac{k^2Mz}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{3(x^2+y^2)-2z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial z}. \end{aligned}$$

The symbols employed are defined as follows: x, y, z are rectangular coördinates, the origin being at the center of the spheroid and the xy-plane the plane of the equator, k is the Gaussian constant, M is the mass of the spheroid, b is the polar radius of the spheroid, μ is the eccentricity of the spheroid,

$$R=\sqrt{x^2+y^2+z^2},\qquad V=\frac{k^2M}{R}\left[1+\frac{b^2}{10}\,\frac{x^2+y^2-2z^2}{R^4}\,\mu^2+\cdots\right].$$

Since

$$\frac{1}{x}\frac{\partial V}{\partial x}=\frac{1}{y}\frac{\partial V}{\partial y},$$

we obtain one integral of areas, namely,

$$(2)\qquad x\frac{dy}{dt}-y\frac{dx}{dt}=c_1.$$

That is, the projection of the area described by the radius vector upon the equatorial plane is proportional to the time. We have also the vis viva integral

$$(3)\qquad \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2V+c_2.$$

For further integration we are compelled to resort to series. Poincaré has shown in his *Les méthodes nouvelles de la mécanique céleste* that if certain

* MOULTON, *Celestial Mechanics*, p. 113.

conditions are fulfilled it is possible to obtain periodic orbits represented by power series in a parameter when periodic orbits are known for zero values of this parameter. These conditions are at least partially fulfilled in the present problem, for the right members of the differential equations are analytic in x, y, z and the parameter μ. Furthermore, for $\mu = 0$, the equations reduce to the ordinary two-body problem for which periodic solutions are known. It is our purpose to show that the remaining necessary conditions also are fulfilled and that periodic solutions persist for values of $\mu \neq 0$. These periodic series are very satisfactory, for the general character of the orbits represented by them is easily obtained. The solutions thus found are rigorous, but they are not general, their existence depending upon special initial conditions.

It will be advantageous to transform the differential equations to cylindrical coördinates by the substitutions

$$
\begin{aligned}
&x = ar\cos v, && k^2 M = n^2 a^3,\\
&y = ar\sin v, && c_1 = ck\sqrt{Ma},\\
&z = aq, && nt = \tau,\\
&R = a\sqrt{r^2+q^2}, && \frac{3}{10}\frac{b^2}{a^2} = \theta_1^2.
\end{aligned}
\tag{4}
$$

After these substitutions equations (1) become

$$
\begin{aligned}
&(a) \quad r'' - rv'^2 = \frac{-r}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{r^2-4q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right],\\
&(b) \quad rv'' + 2r'v' = 0,\\
&(c) \quad q'' = \frac{-q}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{3r^2-2q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right],
\end{aligned}
\tag{5}
$$

the accents denoting the derivatives with respect to τ.

The integral of (b) is $r^2 v' = c$, by means of which v' can be eliminated from the first of these equations. After the elimination the equations take the form

$$
\begin{aligned}
&(a) \quad r'' = \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3-4rq^2}{(r^2+q^2)^{\frac{5}{2}}}\theta_1^2\mu^2 + \cdots,\\
&(b) \quad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2q-q^3}{(r^2+q^2)^{\frac{5}{2}}}\theta_1^2\mu^2 + \cdots,\\
&(c) \quad v' = \frac{c}{r^2}.
\end{aligned}
\tag{6}
$$

The first two of these equations are independent of the third so that r and q may be considered as rectangular coördinates in a revolving plane which passes

always through the z-axis and through the particle itself. The problem is thus reduced to the consideration of the motion in this plane, for, when r is known, v is obtained by a simple quadrature from (6c).

§ 3. *Surfaces of zero velocity.*

The velocity integral in the revolving plane is

$$r'^2 + q'^2 = \frac{2}{(r^2+q^2)^{\frac{1}{2}}} + \frac{2}{3}\frac{r^2-2q^2}{(r^2+q^2)^{\frac{3}{2}}}\theta_1^2\mu^2 + \cdots - \frac{c^2}{r^2} + c_2.$$

If we put the velocity equal to zero the resulting equation represents a two-parameter family of curves. For assigned values of the parameters c and c_2 there is defined a certain curve in the revolving plane. On one side of this curve the motion is real while on the other side it is imaginary. For values of $c_2 < 0$ this curve is closed, and the motion is real on the inside. As the plane revolves this curve generates a surface of the general form of an anchor ring. For $\mu^2 = 0$ this curve belongs to the ordinary two-body problem, and its equation is

$$\frac{2}{(r^2+q^2)^{\frac{1}{2}}} - \frac{c^2}{r^2} + c_2 = 0.$$

The motion is elliptic, parabolic, or hyperbolic according as c_2 is negative, zero, or positive. Putting

$$r = \rho\cos\phi, \qquad q = \rho\sin\phi,$$

we find, on solving for ρ,

$$\rho = \frac{1}{c_2}\left[-1 \pm \sqrt{1 + \frac{c_2c^2}{\cos^2\phi}}\right].$$

For negative values of c_2 this equation represents two closed ovals which do not enclose the origin. If $c_2c^2 = -1$ the ovals shrink upon the points $\rho = -1/c_2$, $\phi = 0$ and π. The corresponding orbit is therefore a circle in the equatorial plane. As c_2 approaches zero the ovals open out rapidly and approach the limiting curves

$$\rho = \frac{c^2}{2\cos^2\phi}.$$

For values of $c_2 > 0$ there is but one positive value for ρ, which is

$$\rho = \frac{1}{c_2}\left[-1 + \sqrt{1 + \frac{c_2c^2}{\cos^2\phi}}\right].$$

If $c^2 \neq 0$ none of these curves crosses the axis $\phi = 90°$. But if $c = 0$ we have the circle $\rho = -2/c_2$ inside of which the motion is real when c_2 is negative.

For values of $\mu^2 \neq 0$, but sufficiently small, we can put

$$r = (\rho + \bar{\rho}) \cos \phi, \qquad q = (\rho + \bar{\rho}) \sin \phi,$$

and solve for $\bar{\rho}$ as a power series in μ^2. We find

$$\bar{\rho} = \frac{1}{3} \frac{2 - 3 \cos^2 \phi}{\rho(1 + c_2 \rho)} \theta_1^2 \mu^2 + \cdots,$$

which is the correction to be applied to the corresponding curves in the two-body problem.

Part I.

Orbits which reënter after one revolution.

§ 4. *Symmetry.*

Returning to the differential equations (6a) and (6b) we observe that if we change

$$r \text{ into } + r, \qquad q \text{ into } - q, \qquad \tau \text{ into } - \tau,$$

the differential equations remain unchanged. Hence, if at some epoch, $\tau = \tau_0$,

$$r = \alpha, \qquad r' = 0,$$

$$q = 0, \qquad q' = \beta,$$

that is, if at the epoch $\tau = \tau_0$, the particle crosses the r-axis perpendicularly, it follows from the form of the differential equations that the orbit is symmetrical with respect to the r-axis and with respect to the epoch $\tau = \tau_0$. In other words r will be an even series in $(\tau - \tau_0)$ and q will be an odd series in $(\tau - \tau_0)$. If now at some other epoch, $\tau = \tau_0 + T$, the particle again crosses the r-axis perpendicularly the orbit will be symmetrical with respect to this epoch also. It is clear therefore that the orbit is a closed one, and that the motion in it is periodic, for, by hypothesis, at $\tau = \tau_0 - T$ it must have been at the same point and moving with the same velocity in the same direction. The motion is therefore periodic with the period $2T$. Hence with these initial values sufficient conditions for periodicity are that at $\tau = \tau_0 + T$

$$r' = q = 0.$$

From the integral of area, $v' = c/r^2$, it follows that if r is periodic v will have the form

$$v = \text{constant} \times \tau + \text{periodic terms}.$$

§ 5. *Existence of periodic orbits in the equatorial plane.*

If $q \equiv 0$ equations (6) reduce to

$$(8)\quad (a)\quad r'' = \frac{c^2}{r^3} - \frac{1}{r^2} - \frac{\theta_1^2\mu^2}{r^4} - \frac{\theta_2^2\mu^4}{r^6} - \cdots,$$

$$(b)\quad v' = \frac{c}{r^2}.$$

The first of these equations is independent of the second and can be integrated separately. It represents motion in a straight line in the revolving plane. It admits the constant solution

$$r_0 = 1, \qquad c_0^2 = 1 + \theta_1^2\mu^2 + \theta_2^2\mu^4 + \cdots,$$

which represents a point in the revolving plane, or a circle in the equatorial plane. In order to investigate the oscillations about this point let us put

$$r = 1 + e\rho, \qquad c^2 = c_0^2 + e\epsilon,$$

where ρ is a variable whose initial value is arbitrarily assigned, e is an arbitrary parameter corresponding to the eccentricity in the two-body problem, and ϵ is a parameter to be determined so that ρ shall be periodic.

On substituting these values (8a) and expanding as power series in e, the terms independent of e cancel out and it is possible to divide through by e. The equation then becomes

$$(9)\quad \begin{aligned} \rho'' + [1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots]\rho &= \epsilon[1 - 3\rho e + 6\rho^2e^2 - 10\rho^3e^3 + \cdots] \\ &\quad + \rho^2 e[3 - 4\theta_1^2\mu^2 - 15\theta_2^2\mu^4 + \cdots] \\ &\quad + \rho^3e^2[-6 + 10\theta_1^2\mu^2 + 46\theta_2^2\mu^4 + \cdots] \\ &\quad + \rho^4e^3[10 - 20\theta_1^2\mu^2 - 111\theta_2^2\mu^4 + \cdots] \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

We can simplify this equation somewhat by dividing through by the coefficient of ρ in the left member and then substituting

$$\mathbf{T} = \tau\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}, \qquad \delta = \frac{\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}.$$

The equation then becomes

$$(10)\quad \begin{aligned} \frac{d^2\rho}{d\mathbf{T}^2} + \rho &= \delta[1 - 3\rho e + 6\rho^2e^2 - 10\rho^3e^3 + \cdots] + [3 + a_1]\rho^2 e \\ &\quad + [-6 + a_2]\rho^3e^2 + [10 + a_3]\rho^4e^3 + \cdots, \end{aligned}$$

where

$$a_1 = -3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots,$$
$$a_2 = +4\theta_1^2\mu^2 + (4\theta_1^4 + 28\theta_2^2)\mu^4 + \cdots,$$
$$a_3 = -10\theta_1^2\mu^2 - (10\theta_1^4 + 81\theta_2^2)\mu^4 + \cdots.$$

Equation (10) can be integrated for ρ as a power series in δ and e. Let us take the initial values

$$\rho = -1, \qquad \rho' = 0.$$

By Poincaré's extension of Cauchy's theorem we know that the solution of equation (10) having the prescribed initial values exists and converges provided δ and e are sufficiently small, for all values of $\mathbf{T}$ in the interval $0 \leqq \mathbf{T} \leqq T$, where T is finite, but otherwise arbitrary. The condition for periodicity is simply

$$\rho' = 0 \text{ at } \mathbf{T} = T. \tag{11}$$

If we choose $T = \pi$ an inspection of equation (10) shows that for $e = 0$ the solution for ρ is periodic with the period 2π whatever may be the value of δ, so that equation (11) must carry e as a factor. After integrating equation (10) we find that the condition (11) is

$$0 = -\left[\tfrac{3}{2} + a_1\right]\pi\delta e - \left[\tfrac{3}{2} + \tfrac{5}{2}a_1 + \tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2\right]\pi e^2 + \text{higher degree terms.} \tag{12}$$

Aside from the factor e there remains an equation in which the linear terms in e and δ are present, and which may be solved. We find

$$\delta = (-1 + \cdots)e + \cdots. \tag{13}$$

If this value of δ be substituted in equation (10) it will then admit periodic solutions for ρ as power series in e having the period 2π for all values of e sufficiently small. Furthermore the solution as a power series in e with the prescribed initial conditions is unique.

§ 6. *Existence of periodic orbits not lying in the equatorial plane.*

For $\mu^2 = 0$ the differential equations (6) admit the circular solution

$$\begin{aligned} r &= 1, & q &= 0, & v &= \tau, \\ r' &= 0, & q' &= 0, & v' &= 1, \end{aligned} \qquad c^2 = 1. \tag{14}$$

In order to investigate the existence of orbits not lying in the equatorial plane but having the period 2π for $\mu^2 \neq 0$, let us put

$$r = 1 + \rho, \qquad q = 0 + \sigma, \qquad c^2 = 1 + \epsilon, \tag{15}$$

and take the initial conditions

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta\mu.$$

The conditions for periodicity (§4) are then

$$\rho' = \sigma = 0 \text{ at } \tau = \pi.$$

We have three arbitrary constants at our disposal, α, β, and ϵ, and two conditions to be satisfied. Hence one constant remains undetermined. We will therefore let β remain arbitrary and determine α and ϵ so as to satisfy the two conditions. After making the substitutions (15) and expanding, equations (6) become

$$\begin{aligned} \rho'' + \rho &= \epsilon - 3\rho\epsilon + 3\rho^2 + \tfrac{3}{2}\sigma^2 - \theta_1^2\mu^2 + 6\rho^2\epsilon - 6\rho^3 - 6\rho\sigma^2 \\ &\quad + 4\rho\theta_1^2\mu^2 + \text{higher degree terms}, \\ \sigma'' + \sigma &= 3\rho\sigma - 6\rho^2\sigma + \tfrac{3}{2}\rho^3 - \sigma\theta_1^2\mu^2 + \text{higher degree terms}. \end{aligned} \tag{16}$$

In order to integrate these equations let us put

$$\begin{aligned} \rho &= \sum_{i,j,k=0}^{\infty} \rho_{ijk}\epsilon^i\alpha^j\mu^k, \\ \sigma &= \sum_{i,j,k=1}^{\infty} \sigma_{ijk}\epsilon^i\alpha^j\mu^k. \end{aligned} \qquad (i+j+k>0), \tag{17}$$

The ρ_{ijk} and σ_{ijk} can be found by successive integrations, the constants of integration being determined so as to satisfy the initial conditions. In the series thus obtained take $\tau = \pi$. The two conditions for periodicity give us the two equations:

$$\begin{aligned} &(a)\ \rho'_{\tau=\pi} = 0 = a_1\epsilon^2 + a_2\epsilon\alpha + a_3\epsilon^3 + a_4\epsilon^2\alpha + a_5\epsilon\alpha^2 + a_6\alpha^3 + a_7\epsilon\mu^2 + a_8\alpha\mu^2 + a_9\mu^4 + \cdots, \\ &(b)\ \sigma_{\tau=\pi} = 0 = \beta\mu\,[\,b_1\epsilon + b_2\epsilon^2 + b_3\alpha^2 + b_4\epsilon\alpha + b_5\mu^2 + \cdots\,], \end{aligned} \tag{18}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero. Equation (18a) involves only the even powers of μ while (18b) involves only the odd powers. After dividing (18b) by $\beta\mu$ we can solve it for ϵ as a power-series in α and μ^2 of the form

$$\epsilon = c_1\alpha^2 + c_2\mu^2 + c_3\alpha^3 + c_4\alpha\mu^2 + c_5\mu^4 + \cdots. \tag{19}$$

On substituting (19) in (18a) we obtain a series of the form

$$(a)\qquad 0 = d_1\alpha\mu^2 + d_2\alpha^3 + d_3\mu^4 + d_4\alpha^2\mu^2 + d_5\alpha^4 + \cdots. \tag{20}$$

If in this equation we make the substitution

$$(b)\qquad \alpha = \left(\gamma - \frac{d_3}{d_1}\right)\mu^2,$$

we obtain

$$(c)\qquad 0 = \mu^4[\,f_1\gamma + f_2\mu^2 + f_3\gamma\mu^2 + \cdots\,],$$

which can be solved uniquely for γ as a power series in μ^2. This solution substituted in (20b) gives α as a power series in μ^2, and this value of α substituted in (19) gives ϵ as a power series in μ^2. We thus have a solution

$$\alpha = \mu^2 P_1(\mu^2), \qquad \epsilon = \mu^2 P_2(\mu^2), \qquad \beta \text{ arbitrary.}$$

Newton's parallelogram shows that equation (20a) has two additional solutions, but as they are imaginary we do not stop to develop them.

§ 7. *Existence of periodic orbits in a meridian plane.*

If in equations (6) we take the area constant $c^2 = 0$ the motion of the particle is in a meridian plane; that is, the revolving plane has ceased to revolve and the orbit in this plane is the true orbit. After changing to polar coördinates by the substitution

$$r = p\cos\phi, \qquad q = p\sin\phi, \qquad c^2 = 0,$$

the differential equations become

$$(21)\qquad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

For $\mu^2 = 0$ we have the periodic solution

$$p = 1, \qquad \phi = \tau,$$

that is, a circle. For $\mu^2 \neq 0$ let us introduce ρ and σ by

$$p = 1 + \rho, \qquad \phi = \tau + \sigma,$$

with the initial values

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta,$$

where α and β are two new arbitraries. By Poincaré's theorem ρ, ρ', σ and σ' are expansible as power series in α, β and μ^2 with τ entering the coefficients. The conditions for periodicity are that at $\tau = \pi$

$$\rho' = \sigma = 0.$$

If we perform the integration and then set $\tau = \pi$, we obtain the two following conditional equations:

$$(22)\qquad \begin{aligned} &(a)\quad \sigma_{\tau=\pi} = 0 = a_1\alpha + a_2\beta + a_3\alpha^2 + a_4\alpha\beta + a_5\beta^2 + a_6\mu^2 + \cdots, \\ &(b)\quad \rho'_{\tau=\pi} = 0 = \qquad\qquad b_3\alpha^2 + b_4\alpha\beta + b_5\beta^2 + 0\cdot\mu^2 + b_7\mu^4 + \cdots, \end{aligned}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero.

The first of these two equations can be solved for α as a power series in β and μ^2. This expression for α substituted in (b) gives rise to an equation of the form

$$(c) \qquad 0 = c_1\beta\mu^2 + c_2\beta^3 + c_3\beta^2\mu^2 + c_4\mu^4 + \cdots.$$

This equation has the same form as $(20a)$ and can be solved in the same way, giving a unique real value for β as a power series in μ^2 vanishing with μ^2. This expression for β substituted in the equation for α gives α as a power series in μ^2, vanishing with μ^2. Therefore real periodic orbits exist for $\mu^2 \neq 0$ which are analytic continuations of circular orbits for $\mu^2 = 0$.

We have thus proved the existence of the three following classes of periodic orbits which have the period 2π, the generating orbits being circles:

I. Orbits lying in the equatorial plane;

II. Orbits not lying in the equatorial plane;

III. Orbits in a meridian plane.

§ 8. *Construction of periodic solutions in the equatorial plane.*

Let us first consider orbits in the equatorial plane. We retake the differential equations (8) and by means of the transformations there given we pass at once to equation (10), which is explicitly

$$(23) \qquad \begin{aligned} \frac{d^2\rho}{d\mathbf{T}^2} + \rho = \delta[1 - 3\rho e + 6\rho^2 e^2 + \cdots] + [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]\rho^2 e \\ + [-6 + 4\theta_1^2\mu^2 + (4\theta_1^2 + 28\theta_2^2)\mu^4 + \cdots]\rho^3 e^2 \\ + [10 - 10\theta_1^2\mu^2 - (10\theta_1^4 + 81\theta_2^2)\mu^4 + \cdots]\rho^4 e^3 + \cdots. \end{aligned}$$

It was shown in equation (13) that δ can be expanded uniquely as a power series in e in such a manner that the solution for ρ as a power series in e will be periodic with the period 2π. Since the series is periodic with the same period for all values of e sufficiently small, it follows that the coefficient of each power of e is itself periodic. Since the solution exists and is unique, it must be possible to determine the δ uniquely by the condition that the solution is periodic. In the existence proof it was shown that δ vanishes with e. Therefore ρ and δ have the form

$$(24) \qquad \rho = \rho_0 + \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \cdots, \qquad \delta = \delta_1 e + \delta_2 e^2 + \delta_3 e^3 + \cdots.$$

The ρ_j are to be determined by the integration of equation (23) and by the initial values

$$\rho = -1, \qquad \frac{d\rho}{d\mathbf{T}} = 0 \text{ at } \mathbf{T} = 0.$$

The δ_j are to be determined in such a manner that the ρ_j shall be periodic.

Substituting (24) in (23) and equating the coefficients, we find:

$$(a)\quad \frac{d^2\rho_0}{d\mathbf{T}^2}+\rho_0=0,$$

$$(b)\quad \frac{d^2\rho_1}{d\mathbf{T}^2}+\rho_1=\delta_1+[\,3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\rho_0^2,$$

$$(25)\quad (c)\quad \frac{d^2\rho_2}{d\mathbf{T}^2}+\rho_2=\delta_2-3\rho_0\delta_1+[\,3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_1 + [\,-6+4\theta_1^2\mu^2+(4\theta_1^4+28\theta_2^2)\mu^4+\cdots]\rho_0^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(d)\quad \frac{d^2\rho_n}{d\mathbf{T}^2}+\rho_n=\delta_n-3\rho_0\delta_{n-1}+[\,3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_{n-1} + f_n(\rho_0,\cdots,\rho_{n-2}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These equations can be integrated in succession. The solution of (a) which satisfies the initial conditions is

$$\rho_0=-\cos\mathbf{T}. \tag{26}$$

Since the initial conditions are independent of e, every ρ_j except ρ_0 must vanish at $\mathbf{T}=0$. Substituting (26) in (25b) and integrating, we have

$$\rho_1=\delta_1(1-\cos\mathbf{T})+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\left[\tfrac{1}{2}-\tfrac{1}{3}\cos\mathbf{T}-\tfrac{1}{6}\cos 2\mathbf{T}\right]. \tag{27}$$

The constants of integration in equation (27) have been determined so as to satisfy the initial conditions, but the constant δ_1 is as yet undetermined.

Substituting these values of ρ_0 and ρ_1 in (25c), we find

$$\begin{aligned}\frac{d^2\rho_2}{d\mathbf{T}^2}+\rho_2 &= [\,\delta_2+\delta_1(3-3\theta_1^2\mu^2+\cdots)+3-6\theta_1^2\mu^2+\cdots]\\ &\quad+[\,\delta_1(-3+6\theta_1^2\mu^2+\{2\theta_1^4+12\theta_2^2\}\mu^4+\cdots)\\ &\quad\qquad+(-3+12\theta_1^2\mu^2-\{\tfrac{11}{2}\theta_1^2-9\theta_2^2\}\mu^4+\cdots]\cos\mathbf{T}\\ &\quad+[\,\delta_1(3-3\theta_1^2\mu^2+\cdots)+3-18\theta_1^2\mu^2+\cdots]\cos 2\mathbf{T}\\ &\quad+[\,3-4\theta_1^2\mu^2+\cdots]\cos 3\mathbf{T}.\end{aligned} \tag{28}$$

In order that the solution of this equation may be periodic the coefficient of $\cos\mathbf{T}$ must be zero. This is the condition which determines δ_1 and consequently

$$\delta_1=-1+2\theta_1^2\mu^2+(\tfrac{3}{2}\theta_1^4-\theta_2^2)\mu^4+\cdots.$$

With this value of δ_1 equation (28) becomes

$$\frac{d^2\rho_2}{d\mathbf{T}^2} + \rho_2 = [\delta_2 + 3\theta_1^2\mu^2 + \cdots] + [-9\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T} + [3 - 4\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T}.$$

The solution of this equation which satisfies the initial conditions is

$$\begin{aligned}(29)\qquad \rho_2 = \delta_2(1 - \cos\mathbf{T}) &+ [3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\tfrac{3}{2}\theta_1^2\mu^2 + \cdots]\cos\mathbf{T} \\ &+ [3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T} + [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T}.\end{aligned}$$

The constant δ_2 is as yet undetermined. It is determined by the periodicity condition for ρ_3 in the same manner that δ_1 was determined by the periodicity condition of ρ_2. Without giving the details of the computation its value is found to be

$$\delta_2 = -6\theta_1^2\mu^2 + \cdots.$$

This method of integration can be carried as far as may be desired. In orderto show this let us suppose that $\rho_0, \cdots, \rho_{n-1}$ have been computed and all the constants determined with the exception of δ_{n-1}. From $(25d)$ we have

$$(30)\qquad \frac{d^2\rho_n}{d\mathbf{T}^2} + \rho_n = \delta_n - 3\rho_0\delta_{n-1} + [3 - 3\theta_1^2\mu^2 + \cdots]2\rho_0\rho_{n-1} + f_n(\rho_0, \cdots, \rho_{n-2}).$$

Here $f_n(\rho_0, \cdots, \rho_{n-2})$ is a polynomial in the ρ_j and contains only known terms. ρ_{n-1} depends upon δ_{n-1} in the following way:

$$\rho_{n-1} = \delta_{n-1}(1 - \cos\mathbf{T}) + \text{known terms}.$$

Equation (30) can therefore be written

$$\begin{aligned}\frac{d^2\rho_n}{d\mathbf{T}^2} + \rho_n = \delta_n + [3 - 3\theta_1^2\mu^2 + \cdots]\delta_{n-1} + [-3 + 6\theta_1^2\mu^2 + \cdots]\delta_{n-1}\cos\mathbf{T} \\ + [3 - 3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T} + \text{known terms}.\end{aligned}$$

In order that the solution of this equation shall be periodic the coefficient of $\cos\mathbf{T}$ must be zero. This condition determines δ_{n-1}. The equation can then be integrated and the constants of integration will be determined by the conditions that

$$\rho_n = \frac{d\rho_n}{d\mathbf{T}} = 0 \text{ at } \mathbf{T} = 0.$$

Everything is then determined with the exception of δ_n, and we have then

$$\rho_n = (1 - \cos\mathbf{T})\delta_n + \text{known terms}.$$

Substituting the values of δ_1 and δ_2 in the solution as far as it has been com-

puted, we find

$$\rho_0 = -\cos \mathbf{T},$$

$$\rho_1 = [\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 \cdots] + [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots]\cos \mathbf{T}$$
$$+ [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 \cdots]\cos 2\mathbf{T},$$

$$\rho_2 = [-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos \mathbf{T} + [3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T}$$
$$+ [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T},$$

$$\delta_1 = -1 + 2\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 - \theta_2^2)\mu^4 + \cdots,$$

$$\delta_2 = -6\theta_1^2\mu^2 + \cdots.$$

From these expressions we have the following series for $r = 1 + e\rho$:

$$\begin{aligned}(a)\quad r = {} & 1 - e\cos \mathbf{T} + \{[\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 + \cdots] \\ & + [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots]\cos \mathbf{T} \\ (31)\qquad & + [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 + \cdots]\cos 2\mathbf{T}\}e^2 \\ & + \{[-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos \mathbf{T} \\ & + [3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T} + [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T}\}e^3 + \cdots.\end{aligned}$$

Substituting this value of r in the equation (8b)

$$r^2\frac{dv}{d\mathbf{T}} = \sqrt{\frac{c_0^2 + e\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 \cdots}},$$

and integrating, we find

$$\begin{aligned}(b)\quad v - v_0 = {} & \{[1 + \theta_1^2\mu^2 + (\tfrac{1}{2}\theta_1^4 + 2\theta_2^2)\mu^4 + \cdots] \\ & + [\theta_1^2\mu^2 + (-\tfrac{9}{4}\theta_1^4 + 1\tfrac{9}{2}\theta_2^2)\mu^4 + \cdots]e^2 + \cdots\}\mathbf{T} \\ (31)\qquad & + \{[2 + 2\theta_1^2\mu^2 + (\theta_1^4 + \theta_2^2)\mu^4 + \cdots]\sin \mathbf{T}\}e \\ & + \{[2\theta_1^2\mu^2 + (1\tfrac{3}{3}\theta_1^4 - 6\theta_2^2)\mu^4 + \cdots]\sin \mathbf{T} \\ & + [\tfrac{5}{4} + \tfrac{3}{4}\theta_1^2\mu^2 + (-2\tfrac{1}{4}\theta_1^4 + \tfrac{3}{2}\theta_2^2)\mu^4 + \cdots]\sin 2\mathbf{T}\}e^2 + \cdots.\end{aligned}$$

Equations (31a) and (31b) are the periodic solutions sought. If we return to the symbols defined in the original differential equations by means of equations (4), with the additional notation

$$n\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots} = \nu,$$

we have the following expressions for the polar coördinates at any time t:

$$R = a\{1 - e\cos\nu t + [\tfrac{1}{2} - \tfrac{1}{2}\cos 2\nu t]e^2 + [\tfrac{3}{8}\cos\nu t - \tfrac{3}{8}\cos 3\nu t]e^3 + \cdots$$

$$+ \frac{b^2}{a^2}\mu^2[(\tfrac{3}{2}\tfrac{3}{0} - \tfrac{3}{1}\tfrac{3}{0}\cos\nu t + \tfrac{3}{2}\tfrac{3}{0}\cos 2\nu t)e^2 + (-\tfrac{9}{1}\tfrac{9}{0} - \tfrac{3}{2}\tfrac{3}{0}\cos\nu t \tag{32}$$

$$+ \tfrac{9}{1}\tfrac{9}{0}\cos 2\nu t + \tfrac{3}{2}\tfrac{3}{0}\cos 3\nu t)e^3 + \cdots] + \frac{b^4}{a^4}\mu^4[\cdots] + \cdots\},$$

$$v - v_0 = \nu t + 2\sin\nu t\cdot e + \tfrac{5}{4}\sin 2\nu t\cdot e^2 + \cdots + \frac{b^2}{a^2}\mu^2[(\tfrac{3}{1}\tfrac{3}{0} + \tfrac{3}{1}\tfrac{3}{0}e^2 + \cdots)\nu t \tag{33}$$

$$+ (\tfrac{3}{5}\sin\nu t)e^2 + (\tfrac{3}{5}\sin\nu t + \tfrac{9}{4}\tfrac{9}{0}\sin 2\nu t)e^2 + \cdots] + \frac{b^4}{a^4}\mu^4[\cdots] + \cdots.$$

These equations contain four arbitrary constants, a, e, v_0 and t_0. Since the differential equations of motion in the equatorial plane were of the fourth order these series, within the realm of their convergence, represent the general solution. The expression for the radius vector, R, is always periodic with the period $2\pi/\nu$. At the expiration of this period v has increased by the quantity

$$2\pi\left[\frac{b^2}{a^2}\mu^2(\tfrac{3}{1}\tfrac{3}{0} + \tfrac{3}{1}\tfrac{3}{0}e^2 + \cdots) + \cdots\right] = 2\pi\Theta$$

in excess of 2π; that is, the line of apsides has rotated forwarded by this amount. If Θ is commensurable with unity the orbit is eventually closed geometrically. If $\Theta = I/J$, where I and J are integers relatively prime, then, at $t = 2J\pi/\nu$, $v = 2(I+J)\pi$, and the particle is at its initial position with its initial components of velocity. The particle has completed $I+J$ revolutions and the line of apsides has completed I revolutions. The mean sidereal period is

$$P = \frac{2\pi}{\nu(1+\Theta)}.$$

This formula for the rotation of the line of apsides has an interesting application in the case of Jupiter's fifth satellite. On the hypothesis that Jupiter is homogeneous and taking its equatorial diameter as 90,190 miles and its polar diameter as 84,570 miles, the mean distance of the satellite as 112,500 miles, the eccentricity of its orbit as .006 and its mean sidereal period as $11^h\,0^m\,23^s$, the above formula gives for the rotation of the line of apsides 1440° per year. The values derived from observations are somewhat discordant but are in the neighborhood of 880° per year. If we still keep the hypothesis that Jupiter is homogeneous in density and of the same oblateness as before, we are compelled, in order to relieve the discrepancy between theory and observation, to suppose that the value adopted for its polar radius was about 9,000 miles too

great. From the large reduction required it is clear that the hypothesis of homogeneity is very much in error.

§ 9. *Construction of periodic solutions not lying in the equatorial plane.*

By means of the area integral the problem has been reduced to the two equations

$$
\begin{aligned}
r'' &= \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3 - 4rq^2}{(r^2+q^2)^{\frac{5}{2}}}\theta_1^2\mu^2 + \cdots, \\
q'' &= -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2q - 2q^3}{(r^2+q^2)^{\frac{5}{2}}}\theta_1^2\mu^2 + \cdots.
\end{aligned} \tag{35}
$$

After the solution of these equations has been obtained the third coördinate is found from the equation

$$v' = \frac{c}{r^2}.$$

We have already proved [equations (14) to (20)] the existence of periodic solutions of these equations of the following type:

$$
\begin{aligned}
r &= 1 + \rho_2\mu^2 + \rho_4\mu^4 + \cdots, \\
q &= q_1\mu + q_3\mu^3 + q_5\mu^5 + \cdots, \\
c^2 &= 1 + c_2\mu^2 + c_4\mu^4 + \cdots,
\end{aligned} \tag{36}
$$

with the initial conditions

$$r'(0) = q(0) = 0, \qquad q'(0) = \beta\mu,$$

β being arbitrary. We can therefore integrate equations (35) so as to satisfy these initial conditions and determine the c_j in such a manner as to render the solution periodic.

Substituting (36) in (35) gives the equations

$$
\begin{aligned}
0 = {} & [\rho_2'' + \rho_2 - \tfrac{3}{2}q_1^2 - c_2 + \theta_1^2]\mu^2 + [\rho_4'' + \rho_4 - 3\rho_2^2 - 3q_1q_3 + 6\rho_2q_1^2 \\
& + \tfrac{15}{8}q_1^4 + (3c_2 - 4\theta_1^2)\rho_2 - \tfrac{15}{2}\theta_1^2q_1^2 - c_4]\mu^4 + \cdots,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
0 = {} & [q_1'' + q_1]\mu + [q_3'' + q_3 - 3\rho_2q_1 - \tfrac{3}{2}q_1^3 + 3\theta_1^2q_1]\mu^3 + [q_5'' + q_5 - 3\rho_2q_3 - \tfrac{9}{2}q_1^2q_3 \\
& + 6\rho_2^2q_1 - 3\rho_4q_1 + \tfrac{15}{2}\rho_2q_1^3 + \tfrac{15}{8}q_1^5 + 3\theta_1^2q_3 - 15\theta_1^2\rho_2q_1 - \tfrac{25}{2}\theta_1^2q_1^3]\mu^5 + \cdots.
\end{aligned} \tag{38}
$$

Equation (37) contains only even powers of μ while equation (38) contains only odd powers. For the integration we have:

Coefficient of μ.

$$q_1'' + q_1 = 0. \tag{39}$$

The solution of this equation satisfying the initial conditions is

$$q_1 = \beta \sin \tau.$$

Coefficient of μ^2.

$$\begin{aligned} \rho_2'' + \rho_2 &= \tfrac{3}{2} q_1^2 + c_2 - \theta_1^2, \\ &= (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) - \tfrac{3}{4}\beta^2 \cos 2\tau. \end{aligned} \tag{40}$$

The solution of this equation which satisfies the assigned initial conditions is

$$\rho_2 = (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) + \alpha_2 \cos \tau + \tfrac{1}{4}\beta^2 \cos 2\tau.$$

The constant c_2 will be determined by the periodicity condition on q_3 to be

$$c_2 = 2\theta_1^2 - \beta^2,$$

and α_2 will be determined by the periodicity condition on ρ_4 to be

$$\alpha_2 = 0.$$

If we anticipate these determinations we have

$$\rho_2 = (\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2 \cos 2\tau.$$

Coefficient of μ^3.

$$\begin{aligned} q_3'' + q_3 &= q_1(3\rho_2 + \tfrac{3}{2}q_1^2 - 3\theta_1^2), \\ &= (3\beta^2 + 3c_2 - 6\theta_1^2)\beta \sin \tau + \tfrac{3}{4}\alpha_2\beta \sin 2\tau. \end{aligned} \tag{41}$$

In order that the solution shall be periodic it is necessary that the coefficient of $\sin \tau$ shall be zero. Therefore

$$c_2 = 2\theta_1^2 - \beta^2.$$

Substituting this value and integrating, we find

$$q_3 = \beta_3 \sin \tau - \tfrac{1}{4}\alpha_2\beta \sin 2\tau.$$

From the initial conditions we must have $q_3'(0) = 0$ and therefore

$$\beta_3 = \tfrac{1}{2}\alpha_2\beta.$$

But it will be shown in the next step that $\alpha_2 = 0$, and consequently

$$q_3 \equiv 0.$$

Coefficient of μ^4.

$$\rho_4'' + \rho_4 = 3\rho_2^2 + 3q_1q_3 - 6\rho_2q_1^2 - \tfrac{15}{8}q_1^4 + (4\theta_1^2 - 3c_2)\rho_2 + \tfrac{15}{2}\theta_1^2q_1^2 + c_4. \tag{42}$$

Before expanding the right member of this equation we will avoid useless labor by first examining the coefficient of $\cos \tau$ which we know must be zero from the periodicity condition. We notice first that terms in the coefficient of $\cos \tau$ can

arise only through terms involving ρ_2 and q_3, and second that all such terms carry α_2 as a factor. Since no other arbitrary enters the coefficient, we must take $\alpha_2 = 0$. It can be shown by induction that the constant of integration α_{2i} (the coefficient of $\cos\tau$) which arises in the integration of ρ_{2i} is determined by the periodicity condition on $\rho_{2(i+1)}$, and further that its value is always zero. The proof is omitted for the sake of brevity.

Substituting the value $\alpha_2 = 0$ in ρ_2 and q_3 and expanding the right member of (42), we find

$$\rho_4'' + \rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{3}{16}\beta^4]\cos 2\tau + \tfrac{15}{64}\beta^4\cos 4\tau.$$

The solution of this equation satisfying the initial conditions and the properties just mentioned is

$$\rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4]\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau.$$

Anticipating the value of c_4 found below, we have

$$\rho_4 = [-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^2]\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau.$$

Coefficient of μ^5.

$$q_5'' + q_5 = 5\rho_2 q_3 + \tfrac{9}{2}q_1^2 q_3 - 6\rho_2^2 q_1 + 3\rho_4 q_1 - \tfrac{15}{2}\rho_2 q_1^2 - \tfrac{15}{8}q_1^5 - 3\theta_1^2 q_3 + 15\theta_1^2\rho_2 q_1 + \tfrac{225}{2}\theta_1^2 q_1^3, \tag{43}$$

$$= [3c_4 + 12\theta_1^4 + 11\theta_1^2\beta^2]\beta\sin\tau - \theta_1^2\beta^3\sin 3\tau.$$

From the periodicity condition we have

$$c_4 = -4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2.$$

Integrating and imposing the initial conditions, we find

$$q_5 = -\tfrac{3}{8}\theta_1^2\beta^3\sin\tau + \tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau.$$

This is sufficient to make evident the general character of the series. The r-equation contains only even multiples of τ, and the q-equation contains only odd multiples. The r-equation contains only even powers of μ and of τ. The q-equation is odd in both these respects. The series are therefore triply even and odd.

Collecting the various coefficients we have the following series:

$$(a)\quad r = 1 + [(\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2\cos 2\tau]\mu^2 + [(-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4) + (-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4)\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau]\mu^4 + \cdots,$$

$$(b)\quad q = [\beta\sin\tau]\mu + [0]\mu^3 + [-\tfrac{3}{8}\theta_1^2\beta^3\sin\tau + \tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau]\mu^5 + \cdots, \tag{44}$$

$$(c)\quad v - v_0 = [1 - \theta_1^2\mu^2 - (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots]\tau + [-\tfrac{1}{4}\beta^2\sin 2\tau]\mu^2 + [(\tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{64}\beta^4\sin 4\tau]\mu^4 + \cdots,$$

$$(d)\quad c^2 = 1 + [2\theta_1^2 - \beta^2]\mu^2 + [-4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2]\mu^4 + \cdots.$$

This solution contains four arbitrary constants, α, β, v_0 and τ_0. As is shown by equation (44c) the nodes regress, the rate of regression being

$$[\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots].$$

The generating orbit of these solutions is a circle in the equatorial plane. A circle having any assigned inclination might have been used, e. g.,

$$(45)\quad r = \sqrt{1 - s^2\sin^2\tau}, \qquad q = s\sin\tau, \qquad v = \tan^{-1}[\sqrt{1 - s^2}\tan\tau],$$

where s is the sine of the inclination. The solution thus obtained would have been identical with (44). If we expand (45) as power series in s and then put $s = \beta\mu$, it will be found that the terms thus obtained are identical with the terms independent of θ_1^2 in the solution which has been worked out. It might therefore be of assistance in the physical interpretation of the constants to put $\beta\mu = s$ in the series (44).

§ 10. *Construction of periodic solutions in a meridian plane.*

When the constant c is zero the motion is in a meridian plane. The equations of motion are (21)

$$(46)\quad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

We have already shown the existence of periodic solutions of these equations as power series in μ^2, which for $\mu = 0$ reduce to the circle $p = 1$, $\phi = \tau$. Let us put then

$$p = 1 + p_2\mu^2 + p_4\mu^4 + \cdots, \qquad \phi = \tau + \phi_2\mu^2 + \phi_4\mu^4 + \cdots.$$

Substituting these expressions in (46), expanding and collecting the coefficients of the various powers of μ, we find:

$$(47)\quad \begin{aligned} 0 = {} & [p_2'' - 3p_2 - 2\phi_2' - \tfrac{3}{4}\theta_1^2 + \tfrac{3}{2}\theta_1^2\cos 2\tau + \tfrac{1}{4}\theta_1^2\cos 4\tau]\mu^2 + [p_4'' - 3p_4 - 2\phi_4' - 2p_2\phi_2' \\ & - \phi_2'^2 + 3p_2^2 + (3 - 6\cos 2\tau - \cos 4\tau)\theta_1^2 p_2 + (-3\sin 2\tau - \sin 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots, \\ 0 = {} & [\phi_2'' + 2p_2' + \tfrac{1}{2}\theta_1^2\sin 2\tau - \tfrac{1}{4}\theta_1^2\sin 4\tau]\mu^2 + [\phi_4'' + 2p_4' + \phi_2'' p_2 + 2p_2'\phi_2' \\ & + (-2\sin 2\tau + \sin 4\tau)\theta_1^2 p_2 + (\cos 2\tau - \cos 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots. \end{aligned}$$

The initial conditions are

$$p'(0) = \phi(0) = 0.$$

Proceeding to the integration we have:

Coefficient of μ^2.

$$(48)\quad \begin{aligned} &(a)\quad p_2'' - 3p_2 - 2\phi_2' = \tfrac{3}{4}\theta_1^2 - \tfrac{3}{2}\theta_1^2\cos 2\tau - \tfrac{1}{4}\theta_1^2\cos 4\tau,\\ &(b)\quad \phi_2'' + 2p_2' = -\tfrac{1}{2}\theta_1^2\sin 2\tau + \tfrac{1}{4}\theta_1^2\sin 4\tau. \end{aligned}$$

By integrating (b) once we have

$$(c)\qquad \phi_2' = -2p_2 + \tfrac{1}{4}\theta_1^2\cos 2\tau - \tfrac{1}{16}\theta_1^2\cos 4\tau + c_1.$$

Substituting this value of ϕ_2' in (a), we find

$$(d)\qquad p_2'' + p_2 = (2c_1 + \tfrac{3}{4}\theta_1^2) - \theta_1^2\cos 2\tau - \tfrac{3}{8}\theta_1^2\cos 4\tau.$$

The integration of this equation gives

$$(e)\qquad p_2 = (2c_1 + \tfrac{3}{4}\theta_1^2) + c_2\sin\tau + c_3\cos\tau + \tfrac{1}{3}\theta_1^2\cos 2\tau + \tfrac{1}{40}\theta_1^2\cos 4\tau.$$

Since $p_2' = 0$ at $\tau = 0$ we must take $c_2 = 0$. Then substituting this value of p_2 in (e) and integrating, we get

$$(f)\quad \phi_2 = (-3c_1 - \tfrac{3}{2}\theta_1^2)\tau - 2c_3\sin\tau - \tfrac{5}{24}\theta_1^2\sin 2\tau - \tfrac{9}{320}\theta_1^2\sin 4\tau + c_4.$$

From the initial conditions ϕ_2 must be zero when $\tau = 0$. Therefore $c_4 = 0$. Since it must also be periodic, $c_1 = -\tfrac{1}{2}\theta_1^2$. All of the constants of integration are now determined except c_3 which will be determined by the periodicity condition on p_4.

The differential equations for p_4 and ϕ_4 are the same as for p_2 and ϕ_2 except in the right members. The process of integration is therefore the same. In the right members only even multiples of τ occur except terms carrying the undetermined c_3 as a factor. In the equation corresponding to $(48d)$ there will be a term in $\cos\tau$ carrying c_3 as a factor. But the integration of this term will be non-periodic unless $c_3 = 0$. Put then $c_3 = 0$; the integration proceeds just as before and the constants are determined in the same manner. This argument will be repeated in the coefficients of μ^6 and so on for all higher powers:* Therefore no odd multiples of τ can occur in the solution. We have, therefore,

$$(49)\quad \begin{aligned} p &= 1 + [-\tfrac{1}{4} + \tfrac{1}{3}\cos 2\tau + \tfrac{1}{40}\cos 4\tau]\theta_1^2\mu^2 + \cdots,\\ q &= \tau + [-\tfrac{5}{24}\sin 2\tau - \tfrac{9}{320}\sin 4\tau]\theta_1^2\mu^2 + \cdots. \end{aligned}$$

* The differential equations arising at the successive steps are of the same type as the first two equations of (32), p. 555, in Professor Moulton's paper on periodic solutions of the problem of three bodies, these Transactions, vol. 7 (1906). Consequently his general formulas for the coefficients of the solutions, equations (42), loc. cit., apply here.

Since the series involve only even multiples of τ the orbits are symmetrical with respect to both the r-axis and the q-axis.

This completes the formal construction of the solutions of which the existence was proved in §§ 5, 6 and 7.

PART II.

ORBITS REËNTRANT ONLY AFTER MANY REVOLUTIONS.

§ 11. *The differential equations.*

The orbits which we have previously considered have had the common property of involving only the period 2π. Since this period is independent of the oblateness of the spheroid the derivation of these orbits has been relatively simple. We shall proceed now to investigate a class of orbits which involves beside the period 2π another period $2\pi/\lambda$, where λ is a function of the oblateness of the spheroid, the inclination of the orbit and the mean distance of the particle. We shall start from the solution which involves an arbitrary inclination. Into this solution four arbitrary constants were introduced, viz., inclination, mean distance, longitude of node and the epoch. Two more arbitraries are necessary for a complete solution, viz., constants corresponding to the eccentricity and to the longitude of perihelion from the node. In what follows we shall introduce the constant corresponding to the eccentricity, but we shall still project the particle from an apse at the node.

We have found for the differential equations a certain solution (44) which we may write

$$r = \phi(\beta, \mu; \tau), \qquad q = \psi(\beta, \mu; \tau), \qquad c^2 = c_0^2,$$

which is symmetric with respect to the equatorial plane. That is to say, at $\tau = 0$ the particle is in the equatorial plane and its motion is perpendicular to the radius vector. Its initial distance is $\phi(0)$. Suppose now we change the initial distance slightly and also the initial velocity so that at $\tau = 0$

$$r(0) = \phi(0) + \alpha, \qquad q(0) = 0,$$

$$r'(0) = 0, \qquad q'(0) = \psi'(0) + \gamma,$$

and give an increment to the constant of areas so that $c^2 = c_0^2 + \epsilon$. Can we determine these three constants, α, γ and ϵ, in such a manner that the series for r and q shall be periodic? These series can be expressed by

$$r = \phi(\beta, \mu; \tau) + \rho, \qquad q = \psi(\beta, \mu; \tau) + \sigma, \qquad c^2 = c_0^2 + \epsilon.$$

If now we substitute these expressions in the differential equations (6) all the terms independent of ρ, σ and ϵ will drop out, and there will remain the fol-

lowing differential equations for ρ and σ:

$$
\begin{aligned}
(a)\ \rho'' &+ \{1 + [(-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2\cos 2\tau]\mu^2 + [(20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) \\
&+ (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4\cos 4\tau]\mu^4 + \cdots\}\rho + \{-3\beta\mu\sin\tau \\
&+ [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3\sin 3\tau]\mu^3 + \cdots\}\sigma \\
&+ \epsilon\Big\{\{1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2\cos 2\tau]\mu^2 + [(15\theta_1^4 - \tfrac{1}{4}\theta_1^2\beta^2 + \tfrac{4\,5}{6\,4}\beta^4) \\
&+ (\tfrac{1\,3}{4}\theta_1^2\beta^2 - \tfrac{1\,5}{1\,6}\beta^4)\cos 2\tau + \tfrac{1\,5}{4}\beta^4\cos 4\tau]\mu^4 + \cdots\} \\
&+ \{-3 + [(12\theta_1^2 - 3\beta^2) + 3\beta^2\cos 2\tau]\mu^2 + \cdots\}\rho + \cdots\Big\} \\
&= \{3 + [(-16\theta_1^2 + 6\beta^2) - 12\beta^2\cos 2\tau]\mu^2 + [(90\theta_1^4 + \tfrac{5\,9}{4}\theta_1^2\beta^2 + \tfrac{6\,9}{6\,4}\beta^4) \\
&+ (\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{6\,3}{1\,6}\beta^4)\cos 2\tau + \tfrac{1\,3}{6\,4}\beta^4\cos 4\tau]\mu^4 + \cdots\}\rho^2 + \{-12\beta\mu\sin\tau \\
&+ [(-90\theta_1^2\beta + \tfrac{4\,5}{4}\beta^3)\sin\tau - \tfrac{1\,5}{4}\beta^3\sin 3\tau]\mu^3 + \cdots\}\rho\sigma \\
&+ \{\tfrac{3}{2} + [(\tfrac{3}{2}\theta_1^2 - \tfrac{3\,3}{8}\beta^2) + \tfrac{3\,3}{8}\beta^2\cos 2\tau]\mu^2 + \cdots\}\sigma^2 + \\
&\quad\cdots\cdots\cdots

These equations admit a solution of the form *

$$\rho_j = A_j e^{\lambda_j \tau} \phi_j(\tau),$$

$$\sigma_j = A_j e^{\lambda_j \tau} \psi_j(\tau) \qquad (j = 1, \cdots, 4)$$

where λ_j is a root of the fundamental equation and where $\phi_j(\tau)$ and $\psi_j(\tau)$ are periodic functions of τ with the period 2π. The four values of the λ_j (real or imaginary) are associated in pairs, equal in value but of opposite sign.† From the fact that τ does not occur explicitly in the original equations (1) it is known *à priori* that one pair of the λ_j has the value 0,‡ and consequently, if we choose the notation so that $\lambda_3 = \lambda_4 = 0$, the two corresponding solutions will have the forms

$$\rho_3 = A_3 \phi_3(\tau), \qquad \rho_4 = A_4[\phi_4(\tau) + \tau\phi_3(\tau)],$$

$$\sigma_3 = A_3 \psi_3(\tau), \qquad \sigma_4 = A_4[\psi_4(\tau) + \tau\psi_3(\tau)].$$

We shall consider first the two solutions in which the λ_j are not zero. Let us first substitute in (51) the forms

$$\rho = e^{i\lambda\tau}\phi(\tau) \qquad (i = \sqrt{-1}),$$

$$\sigma = e^{i\lambda\tau}\psi(\tau).$$

After dividing out the exponential there remains

$$(52)\quad \begin{aligned} &\phi'' + 2i\lambda\phi' + [1 - \lambda^2 + a_2\mu^2 + a_4\mu^4 + \cdots]\phi + [a_1\mu + a_3\mu^3 + \cdots]\psi = 0, \\ &\psi'' + 2i\lambda\psi' + [1 - \lambda^2 + b_2\mu^2 + b_4\mu^4 + \cdots]\psi + [a_1\mu + a_3\mu^3 + \cdots]\phi = 0, \end{aligned}$$

where

$$a_1 = -3\beta \sin\tau,$$

$$a_3 = (-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau,$$

$$a_2 = (-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau,$$

$$a_4 = (20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau,$$

$$b_2 = -\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2 \cos 2\tau,$$

$b_4 =$ a sum of cosines of even multiples of τ.

With respect to equations (52) it is known that ϕ and ψ are periodic with the

* Floquet, Annales Scientifiques de l'École Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

† See *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1, p. 193.

‡ Ibid., p. 187.

period 2π, and that λ vanishes with μ since the problem then reduces to the two-body problem in which the characteristic exponents are all zero. It can be shown that ϕ, ψ and λ are expansible as power series in μ of the form

$$\phi = \sum_{j=0}^{\infty} \phi_j \mu^j, \qquad \psi = \sum_{j=0}^{\infty} \psi_j \mu^j, \qquad \lambda = \sum_{j=1}^{\infty} \lambda_j \mu^j.$$

Substituting these expressions in (52) we find:

Coefficient of μ^0.

$$\phi_0'' + \phi_0 = 0,$$

$$\psi_0'' + \psi_0 = 0.$$

Therefore

$$\begin{aligned} \phi_0 &= \alpha_1^{(0)} \cos\tau + \alpha_2^{(0)} \sin\tau, \\ \psi_0 &= \gamma_1^{(0)} \cos\tau + \gamma_2^{(0)} \sin\tau. \end{aligned} \tag{53}$$

Coefficient of μ.

$$\begin{aligned} \phi_1'' + \phi_1 &= -2i\lambda_1\phi_0' - a_1\psi_0, \\ \psi_1'' + \psi_1 &= -2i\lambda_1\psi_0' - a_1\phi_0. \end{aligned} \tag{54}$$

Since the periodicity conditions demand that the coefficients of $\cos\tau$ and $\sin\tau$ in the right members be zero, we must take $\lambda_1 = 0$. We get then

$$\begin{aligned} \phi_1'' + \phi_1 &= \tfrac{3}{2}\beta\gamma_2^{(0)} - \tfrac{3}{2}\beta\gamma_2^{(0)}\cos 2\tau + \tfrac{3}{2}\beta\gamma_1^{(0)}\sin 2\tau, \\ \psi_1'' + \psi_1 &= \tfrac{3}{2}\beta\alpha_2^{(0)} - \tfrac{3}{2}\beta\alpha_2^{(0)}\cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(0)}\sin 2\tau. \end{aligned} \tag{55}$$

Integrating we have

$$\begin{aligned} \phi_1 &= \alpha_1^{(1)}\cos\tau + \alpha_2^{(1)}\sin\tau + \tfrac{3}{2}\beta\gamma_2^{(0)} + \tfrac{1}{2}\beta\gamma_2^{(0)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(0)}\sin 2\tau, \\ \psi_1 &= \gamma_1^{(1)}\cos\tau + \gamma_2^{(1)}\sin\tau + \tfrac{3}{2}\beta\alpha_2^{(0)} + \tfrac{1}{2}\beta\alpha_2^{(0)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(0)}\sin 2\tau. \end{aligned} \tag{56}$$

Coefficient of μ^2.

$$\begin{aligned} \phi_2'' + \phi_2 &= -2i\lambda_2\phi_0' - a_2\phi_0 - a_1\psi_1, \\ \psi_2'' + \psi_2 &= -2i\lambda_2\psi_0' - b_2\psi_0 - a_1\phi_1; \end{aligned} \tag{57}$$

or, expanded,

$$\begin{aligned} \phi_2'' + \phi_2 &= \tfrac{3}{2}\beta\gamma_2^{(1)} - \tfrac{3}{2}\beta\gamma_2^{(1)}\cos 2\tau + \tfrac{3}{2}\gamma_1^{(1)}\sin 2\tau + (4\theta_1^2\alpha_2^{(0)} + 2i\lambda_2\alpha_1^{(0)})\sin\tau \\ &\quad + (4\theta_1^2\alpha_1^{(0)} - 2i\lambda_2\alpha_2^{(0)})\cos\tau + 3\beta^2\alpha_2^{(0)}\sin 3\tau + 3\beta^2\alpha_1^{(0)}\cos 3\tau, \\ \psi_2 + \psi_2 &= \tfrac{3}{2}\beta\alpha_2^{(1)} + (\tfrac{15}{2}\beta^2\gamma_2^{(0)} + 2i\lambda_2\gamma_1^{(0)})\sin\tau + 2i\lambda_2\gamma_2^{(0)}\cos\tau \\ &\quad + \tfrac{3}{2}\beta\alpha_1^{(1)}\sin 2\tau - \tfrac{3}{2}\beta\alpha_2^{(1)}\cos 2\tau. \end{aligned} \tag{58}$$

In order to satisfy the periodicity condition we must have

$$
\begin{aligned}
&2i\lambda_2\alpha_1^{(0)} + 4\theta_1^2\alpha_2^{(0)} = 0, \qquad && 2i\lambda_2\gamma_1^{(0)} + \tfrac{15}{2}\beta^2\gamma_2^{(0)} = 0,\\
&4\theta_1^2\alpha_1^{(0)} - 2i\lambda_2\alpha_2^{(0)} = 0, && + 2i\lambda_2\gamma_2^{(0)} = 0.
\end{aligned} \tag{59}
$$

The equations of the second column are satisfied by taking

$$\gamma_1^{(0)} = \gamma_2^{(0)} = 0.$$

Solving the other two we find

$$\lambda_2 = \pm 2\theta_1^2, \qquad \alpha_1^{(0)} - i\alpha_2^{(0)} = 0.$$

Equations (59) can also be satisfied by taking

$$\lambda_2 = \alpha_1^{(0)} = \alpha_2^{(0)} = \gamma_2^{(0)} = 0, \qquad \gamma_1^{(0)} = \text{arbitrary},$$

but this would lead to the development of the solutions in which the characteristic exponent is zero, and these solutions will be discussed later.

It was known at the outset that there were two values of λ equal numerically but of opposite sign. We will choose the one with the positive sign. The solution for the negative λ can be derived from the solution for the positive λ. The condition $\alpha_1^{(0)} - i\alpha_2^{(0)} = 0$ still leaves us with an arbitrary constant. Since the equations are linear this constant will enter the solution linearly and may therefore be taken equal to unity. The arbitrary constant is restored in (79) after the solutions are completely developed. We will take then $\alpha_1^{(0)} = 1$ which makes $\alpha_2^{(0)} = -i$. Consequently

$$\phi_0 = \cos\tau - i\sin\tau, \qquad \psi_0 = 0. \tag{60}$$

Integrating (58) with these values, we get

$$
\begin{aligned}
\phi_2 = {}& \tfrac{3}{2}\beta\gamma_2^{(1)} + \alpha_1^{(2)}\cos\tau + \alpha_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(1)}\sin 2\tau\\
&+ 3\beta^2\cos 3\tau - 3\beta^2 i\sin 3\tau,\\
\psi_2 = {}& \tfrac{3}{2}\beta\alpha_2^{(1)} + \gamma_1^{(2)}\cos\tau + \gamma_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\alpha_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(1)}\sin 2\tau.
\end{aligned} \tag{61}
$$

Coefficient of μ^3.

$$
\begin{aligned}
\phi_3'' + \phi_3 = {}& \tfrac{3}{2}\beta\gamma_2^{(2)} + [-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)})]\cos\tau\\
&+ [2i\lambda_3 + 4\theta_1^2(\alpha_2^{(1)} + i\alpha_1^{(1)})]\sin\tau + \tfrac{3}{2}\beta\gamma_1^{(2)}\sin 2\tau - \tfrac{3}{2}\beta\gamma_2^{(2)}\cos 2\tau,\\
\psi_3'' + \psi_3 = {}& [\tfrac{5}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + [-2i\lambda_2\gamma_2^{(1)} + \tfrac{3}{4}\beta^2\gamma_1^{(1)}]\cos\tau\\
&+ [2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)}]\sin\tau + [-\tfrac{3}{2}\alpha_2^{(2)}\beta + \tfrac{11}{2}i\theta_1^2\beta + \tfrac{21}{16}i\beta^3]\cos 2\tau\\
&+ [\tfrac{3}{2}\alpha_1^{(2)}\beta + \tfrac{11}{2}\theta_1^2\beta - \tfrac{9}{16}\beta^3]\sin 2\tau - \tfrac{3}{4}\beta^2\gamma_1^{(1)}\cos 3\tau - \tfrac{3}{4}\beta^2\gamma_2^{(1)}\sin 3\tau.
\end{aligned} \tag{62}
$$

From the periodicity conditions we must have

$$-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0, \qquad \tfrac{3}{4}\beta^2\gamma_1^{(1)} - 2i\lambda_2\gamma_2^{(1)} = 0,$$
$$2i\lambda_3 + 4\theta_1^2(i\alpha_1^{(1)} + \alpha_2^{(1)}) = 0, \qquad 2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)} = 0.$$

The last two equations can be satisfied only if $\gamma_1^{(1)} = \gamma_2^{(1)} = 0$. The first two can be satisfied only if $\lambda_3 = (\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$. The condition $(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$ again gives us an arbitrary constant. Then by (56) $\phi_1 = c(\cos\tau - i\sin\tau)$, but this is the same as ϕ_0 multiplied by $c\mu$. That is, the solution is repeating itself one degree higher in μ and this, of course, should be expected since the equations are linear, so that any solution multiplied by any power of μ must satisfy them. We are at liberty then to choose the arbitrary $c = 0$, which is the same as choosing $\alpha_1^{(1)} = \alpha_2^{(1)} = 0$. Integrating (62) with these values, we find

$$\phi_3 = \tfrac{3}{2}\beta\gamma_2^{(2)} + \alpha_1^{(3)}\cos\tau + \alpha_2^{(3)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(2)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(2)}\sin 2\tau,$$

$$\psi_3 = [\tfrac{3}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + \gamma_1^{(3)}\cos\tau + \gamma_2^{(3)}\sin\tau$$
$$+ [\tfrac{1}{2}\alpha_2^{(2)}\beta - \tfrac{11}{6}i\theta_1^2\beta - \tfrac{7}{16}i\beta^3]\cos 2\tau + [-\tfrac{1}{2}\alpha_1^{(2)}\beta - \tfrac{11}{6}\theta_1^2\beta + \tfrac{3}{16}\beta^3]\sin 2\tau. \tag{63}$$

It can be shown by induction at this point that ϕ and λ involve only even powers of μ, and that ψ is an odd series in μ. Furthermore ϕ contains only odd multiples of τ, and ψ only even multiples. Consequently

$$\gamma_1^{(2)} = \gamma_2^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = \gamma_1^{(3)} = \gamma_2^{(3)} = 0,$$

and all

$$\phi_{2j+1} = \psi_{2j} = 0.$$

Coefficient of μ^4.

$$\phi_4'' + \phi_4 = [-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2]\cos\tau + [2i\lambda_4$$
$$+ 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4]\sin\tau + [6\theta_1^2\beta^2 + 3\beta^2\alpha_1^{(2)} + \tfrac{3}{16}\beta^4]\cos 3\tau$$
$$+ [-6i\theta_1^2\beta^2 + 3\beta^2\alpha_2^{(2)} - \tfrac{45}{16}i\beta^4]\sin 3\tau - \tfrac{21}{16}\beta^4\cos 5\tau + \tfrac{21}{16}i\beta^4\sin 5\tau. \tag{64}$$

Therefore, from the periodicity condition, we must have

$$-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2 = 0,$$
$$2i\lambda_4 + 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4 = 0.$$

Solving these equations, we find

$$\lambda_4 = -8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2, \qquad \alpha_1^{(2)} - i\alpha_2^{(2)} = \tfrac{5}{4}\beta^2.$$

In this last equation we can choose $\alpha_1^{(2)} = \frac{5}{4}\beta^2$ and $\alpha_2^{(2)} = 0$. This choice of $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ will make the coefficient of $\sin\tau$ in ϕ_2 equal to zero, and since the same thing occurs for each ϕ_j it is evident that this method of choosing will simplify

the solution by making the coefficient of $\sin\tau$ equal to zero for all powers of μ. We have then on integrating

$$\begin{aligned}
\phi_4 &= \alpha_4\cos\tau + [-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4]\cos 3\tau + i[\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4]\sin 3\tau \\
&\qquad + \tfrac{7}{128}\beta^4\cos 5\tau - \tfrac{7}{128}i\beta^4\sin 5\tau, \\
\psi_3 &= i[-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3] + i[-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3]\cos 2\tau + [-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3]\sin 2\tau, \\
\phi_2 &= \tfrac{5}{4}\beta^2\cos\tau - \tfrac{3}{8}\beta^2\cos 3\tau + \tfrac{3}{8}i\beta^2\sin 3\tau, \\
\psi_1 &= -\tfrac{3}{2}i\beta - \tfrac{1}{2}i\beta\cos 2\tau - \tfrac{1}{2}\beta\sin 2\tau, \\
\phi_0 &= \cos\tau - i\sin\tau, \\
\lambda &= 2\theta_1^2\mu^2 + (-8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2)\mu^4 + \cdots.
\end{aligned} \tag{65}$$

The coefficient α_4 of $\cos\tau$ in ϕ_4 is determined by the periodicity condition on ϕ_6. That the process of determining the values of the λ_j and the constants of integration arising at each step is general may be shown as follows. Let us suppose that we have computed everything up to and including ϕ_j with the exception of the constants of integration in ϕ_j. We have then

$$\phi_j = \alpha_1^{(j)}\cos\tau + \alpha_2^{(j)}\sin\tau + \text{known terms}.$$

The $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$ enter ψ_{j+1} as follows:

$$\begin{aligned}
\psi_{j+1}'' + \psi_{j+1} &= 3\beta\sin\tau\,\phi_j + \text{known terms}, \\
&= \tfrac{3}{2}\beta\alpha_2^{(j)} - \tfrac{3}{2}\beta\alpha_2^{(j)}\cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(j)}\sin 2\tau + \text{known terms}.
\end{aligned}$$

Consequently in so far as it involves $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$

$$\psi_{j+1} = \tfrac{3}{2}\beta\alpha_2^{(j)} + \tfrac{1}{2}\beta\alpha_2^{(j)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(j)}\sin 2\tau.$$

Similarly, in so far as ϕ_{j+2} depends upon constants as yet undetermined

$$\begin{aligned}
\phi_{j+2}'' + \phi_{j+2} &= -2i\lambda_2\phi_j' - 2i\lambda_{j+2}\phi_0' + [(4\theta_1^2 - \tfrac{3}{2}\beta^2) + \tfrac{9}{2}\beta^2\cos 2\tau]\phi_j + 3\beta\sin\tau\,\psi_{j+1} \\
&= [-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2}]\cos\tau \\
&\quad + [2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2}]\sin\tau \\
&\quad + 3\beta^2\alpha_1^{(j)}\cos 3\tau + 3\beta^2\alpha_2^{(j)}\sin 3\tau,
\end{aligned} \tag{66}$$

where A_{j+2} and B_{j+2} are the known terms in the coefficients of $\cos\tau$ and $\sin\tau$ respectively. From the periodicity condition we must have

$$\begin{aligned}
-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2} &= 0, \\
+2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2} &= 0.
\end{aligned} \tag{67}$$

The solution of these equations is

$$\lambda_{j+2} = \tfrac{1}{4}(A_{j+2} + iB_{j+2}),$$
$$\alpha_1^{(1)} - i\alpha_2^{(j)} = -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}. \tag{68}$$

As has already been pointed out we can choose $\alpha_1^{(j)} = 0$ and we have then

$$\alpha_2^{(j)} = -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}. \tag{69}$$

In order to show that λ_{j+2} and $\alpha_1^{(j)}$ are real it will be sufficient to show that A_{j+2} is real and B_{j+2} is a pure imaginary. This is readily shown by induction, for up to $j = 4$ inclusive we have

$$\phi_j = \sum_\kappa m_\kappa \cos \kappa\tau + i \sum_\kappa n_\kappa \sin \kappa\tau,$$
$$\psi_j = i \sum_\kappa f_\kappa \cos \kappa\tau + \sum_\kappa g_\kappa \sin \kappa\tau,$$

where m_κ, n_κ, f_κ and g_κ are all real. From the form of the differential equations it follows at once that the same forms hold for $j = 5$, then $j = 6$, and so on. That is A_{j+2} is real while B_{j+2} is a pure imaginary.

It is further to be noticed that A_{j+2} and B_{j+2} do not contain any terms in β independent of θ_1^2, and consequently the θ_1^2 which appears in the denominator of $\alpha_1^{(j)}$ will divide out. This is proved as follows. If θ_1^2 be put equal to zero in the differential equations then equations (52) become the equations of variation of a circular orbit in the ordinary two-body problem, the plane of the circle being inclined to the plane of reference by an angle whose sine is $\beta\mu = s$. The original differential equations (6) can then be written

$$r'' = \frac{(1-s^2)(1-e^2)}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} = R, \quad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} = Q, \tag{70}$$

where the constant c^2 is given the form $(1-s^2)(1-e^2)$. For these equations we have the solution

$$r = a\frac{(1-e^2)\sqrt{1 - s^2\sin^2(\theta-\Omega)}}{1 + e\cos(\theta-\theta_0)},$$
$$q = a\frac{(1-e^2)s\sin(\theta-\Omega)}{1 + e\cos(\theta-\theta_0)}, \tag{71}$$

where

$$(\theta - \theta_0) = (\tau - \tau_0) + 2e\sin(\tau - \tau_0) + \cdots.$$

Let us now form the equations of variation by putting

$$r = r_0 + \rho, \qquad q = q_0 + \sigma, \qquad e = e_0 + \epsilon,$$

where r_0, q_0 and e_0 are the values in (71). We find

$$\begin{aligned} \rho'' &= \frac{\partial R}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial R}{\partial e}\epsilon, \\ \sigma'' &= \frac{\partial Q}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial Q}{\partial e}\epsilon. \end{aligned} \tag{72}$$

Three solutions of these equations are given by *

$$\begin{aligned} \rho &= c_1 \frac{\partial r_0}{\partial \Omega}, & \rho &= c_2 \frac{\partial r_0}{\partial \tau_0}, & \rho &= c_3 \frac{\partial r_0}{\partial e_0}, \\ \sigma &= c_1 \frac{\partial q_0}{\partial \Omega}, & \sigma &= c_2 \frac{\partial q_0}{\partial \tau_0}, & \sigma &= c_3 \frac{\partial q_0}{\partial e_0}, \\ \epsilon &= c_1 \frac{\partial e_0}{\partial \Omega}, & \epsilon &= c_2 \frac{\partial e_0}{\partial \tau_0}, & \epsilon &= c_3 \frac{\partial e_0}{\partial e_0}. \end{aligned} \tag{73}$$

If $e_0 \neq 0$ these three solutions are distinct, but the case in which we are interested is when $e_0 = 0$. In this case it is not difficult to see that the first two solutions coincide. Since the equations are linear, the system

$$\begin{aligned} \rho &= c_4 \left[\frac{\partial r_0}{\partial \Omega} - \frac{\partial r_0}{\partial \tau_0} \right], \\ \sigma &= c_4 \left[\frac{\partial q_0}{\partial \Omega} - \frac{\partial q_0}{\partial \tau_0} \right], \\ \epsilon &= c_4 \left[\frac{\partial e_0}{\partial \Omega} - \frac{\partial e_0}{\partial \tau_0} \right], \end{aligned} \tag{74}$$

is also a solution, but as it vanishes for $e_0 = 0$ it carries e_0 as a factor. We can divide out this factor and absorb it into the arbitrary c_4. For $e_0 = 0$ this solution does not now vanish, and it is moreover distinct from the first solution. Thus we have three distinct solutions even when $e_0 = 0$, but since $\partial Q/\partial e_0 \equiv 0$, and $\partial R/\partial e_0$ carries e_0 as a factor, equations (72) pass over to the equations of variation of a circle when $e_0 = 0$. For these equations we have three solutions which are periodic with the period 2π. The fourth solution is not periodic but involves a term of the form τ times a periodic function.

Let us return now to the solution which we have developed, (65), and consider only the terms which belong to the two-body problem, viz., the terms which are

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 163.

independent of θ_1^2. This solution may be separated into two solutions, one of which is real, the other a pure imaginary. The real solution is the third solution of (73), and the purely imaginary is the second solution. Since both of these solutions are certainly periodic with the period 2π, it follows that no term in β alone can occur in the A_{j+2} and B_{j+2} of (66) for the presence of such terms would give rise to non-periodic terms in the two-body problem. Hence A_{j+2} and B_{j+2} carry θ_1^2 as a factor which may be divided out in equation (69). Further λ_{j+2}, (68), carries θ_1^2 as a factor and therefore λ vanishes with the oblateness of the spheroid.

The solution (65) may now be written

$$\begin{aligned} \rho^{(1)} &= e^{i\lambda\tau}[\phi^{(1)} + i\phi^{(2)}], \\ \sigma^{(1)} &= e^{i\lambda\tau}[\psi^{(1)} + i\psi^{(2)}], \end{aligned} \tag{75}$$

where

$$\begin{aligned} \phi^{(1)} &= \cos\tau + [\tfrac{5}{4}\beta^2\cos\tau - \tfrac{3}{8}\beta^2\cos 3\tau]\mu^2 \\ &\quad + [\alpha_4\cos\tau + (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4)\cos 3\tau + \tfrac{7}{128}\beta^4\cos 5\tau]\mu^4 + \cdots, \\ \phi^{(2)} &= -\sin\tau + [\tfrac{3}{8}\beta^2\sin 3\tau]\mu^2 \\ &\quad + [(\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4)\sin 3\tau - \tfrac{7}{128}\beta^4\sin 5\tau]\mu^4 + \cdots, \\ \psi^{(1)} &= [-\tfrac{1}{2}\beta\sin 2\tau]\mu + [(-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\sin 2\tau]\mu^3 + \cdots, \\ \psi^{(2)} &= [-\tfrac{3}{2}\beta - \tfrac{1}{2}\beta\cos 2\tau]\mu \\ &\quad + [(-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3) + (-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\cos 2\tau]\mu^3 + \cdots. \end{aligned} \tag{76}$$

By putting

$$e^{i\lambda\tau} = \cos\lambda\tau + i\sin\lambda\tau$$

we can write

$$\begin{aligned} \rho^{(1)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] + i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(1)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] + i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned} \tag{77}$$

We have thus one solution of the differential equations. A second solution can be derived from it by merely changing the sign of i. Thus

$$\begin{aligned} \rho^{(2)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] - i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(2)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] - i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned} \tag{78}$$

By adding and subtracting these two solutions we have finally

$$\begin{aligned} \rho &= A[\rho^{(1)} + \rho^{(2)}] + B[\rho^{(1)} - \rho^{(2)}], \\ \sigma &= A[\sigma^{(1)} + \sigma^{(2)}] + B[\sigma^{(1)} - \sigma^{(2)}], \end{aligned} \tag{79}$$

A and B being two arbitrary constants. As above developed there is a certain arbitrariness in these solutions owing to the manner in which the constants of integration were determined. They may be reduced to a normal form by multiplying each solution by the proper power series in μ^2 with constant coefficients. By this process we can make for $\tau = 0$,

$$\rho^{(1)}(0) + \rho^{(2)}(0) = 1, \qquad \sigma^{(1)}(0) - \sigma^{(2)}(0) = \beta\mu.$$

Since $[\rho^{(1)} - \rho^{(2)}]$ and $[\sigma^{(1)} + \sigma^{(2)}]$ are sine series they vanish for $\tau = 0$.

The third and fourth solutions of the equations of variation, (51), are given by *

$$(80) \qquad \begin{aligned} \rho_3 &= C\frac{\partial r_0}{\partial \tau_0}, & \rho_4 &= D\frac{\partial (ar_0)}{\partial a}, \\ \sigma_3 &= C\frac{\partial q_0}{\partial \tau_0}, & \sigma_4 &= D\frac{\partial (aq_0)}{\partial a}. \end{aligned}$$

In forming the partial derivatives with respect to a in the fourth solution it should be remembered that θ_1^2 is an explicit function of a, by equations (4), and that τ is also a function of a implicitly through n. The third solution also can be normalized by giving the arbitrary constant such a form that at $\tau = 0$

$$\rho_3 = 0, \qquad \sigma_3 = C\beta\mu.$$

The fourth solution is non-periodic and has the form

$$(81) \qquad \rho = D\left[\tau\frac{\partial r_0}{\partial \tau_0} + \phi_4\right], \qquad \sigma = D\left[\tau\frac{\partial q_0}{\partial \tau_0} + \psi_4\right],$$

where ϕ_4 and ψ_4 are periodic functions of τ with the period 2π. As in the previous solutions this may be normalized so that at $\tau = 0$

$$\rho_4 = D\beta^2\mu^2, \qquad \sigma_4 = 0.$$

It is also easy to find ϕ_4 and ψ_4 by substituting (81) in the equations of variation and solving for these variables, which must be periodic.

Carrying out the above operations we find the following general solutions

$$\begin{aligned} \rho = A\{&\cos(1-\lambda)\tau + [-\tfrac{1}{4}\beta^2\cos(1-\lambda)\tau + \tfrac{5}{8}\beta^2\cos(1+\lambda)\tau - \tfrac{3}{8}\beta^2\cos(3-\lambda)\tau]\mu^2 \\ &+ [(-\tfrac{1}{2}\alpha_4 + \tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{21}{32}\beta^4)\cos(1-\lambda)\tau + (\tfrac{1}{2}\alpha_4 - \tfrac{365}{64}\beta^4)\cos(1+\lambda)\tau \\ &+ (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4)\cos(3-\lambda)\tau - \tfrac{9}{128}\beta^4\cos(3+\lambda)\tau \\ &+ \tfrac{7}{128}\beta^4\cos(5-\lambda)\tau]\mu^4 + \cdots\} \end{aligned}$$

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 163.

$$
\begin{aligned}
(82)\quad & + B\{\tfrac{1}{2}\sin(1-\lambda)\tau + [(-\tfrac{1}{8}\beta^2 - \tfrac{5}{6}\theta_1^2)\sin(1-\lambda)\tau - \tfrac{5}{16}\beta^2\sin(1+\lambda)\tau \\
& \qquad - \tfrac{3}{16}\beta^2\sin(3-\lambda)\tau]\mu^2 \cdots\} \\
& + C\{[-\tfrac{1}{2}\beta^2\sin 2\tau]\mu^2 + [(\tfrac{1}{16}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{16}\beta^4\sin 4\tau]\mu^4 + \cdots\} \\
& + D\Big|\{[\tfrac{3}{4}\beta^2 + \tfrac{1}{4}\beta^2\cos 2\tau]\mu^2 + [(\tfrac{9}{8}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4) \\
& \qquad + (-\tfrac{9}{8}\theta_1^2\beta^2 + \tfrac{1}{8}\beta^4)\cos 2\tau - \tfrac{1}{32}\beta^4\cos 4\tau]\mu^4 \cdots\} + \tau\{[\tfrac{3}{4}\beta^4\sin 2\tau]\mu^4 \\
& \qquad + [(\tfrac{91}{16}\theta_1^2\beta^4 + \tfrac{15}{64}\beta^6)\sin 2\tau - \tfrac{3}{32}\beta^6\sin 4\tau]\mu^6 + \cdots\}\Big|, \\
\sigma = A & \{[\tfrac{3}{2}\beta\sin\lambda\tau - \tfrac{1}{2}\beta\sin(2-\lambda)\tau]\mu + [\tfrac{3}{2}\theta_1^2\beta\sin\lambda\tau \\
& \qquad - \tfrac{11}{16}\theta_1^2\beta\sin(2-\lambda)\tau]\mu^3 + \cdots\} \\
& + B\{[\tfrac{3}{4}\beta\cos\lambda\tau + \tfrac{1}{4}\beta\cos(2-\lambda)\tau]\mu + [-\tfrac{1}{2}\theta_1^2\beta\cos\lambda\tau \\
(83)\quad & \qquad + \tfrac{1}{2}\theta_1^2\beta\cos(2-\lambda)\tau]\mu^3 + \cdots\} \\
& + C\{[\beta\cos\tau]\mu + [0]\mu^3 + \cdots\} \\
& + D\Big|\{[\tfrac{1}{2}\beta\sin\tau]\mu + [(-\tfrac{7}{4}\theta_1^2\beta + \tfrac{1}{2}\beta^3)\sin\tau]\mu^3 + \cdots\} \\
& \qquad + \tau\{[-\tfrac{3}{2}\beta^3\cos\tau]\mu^3 + [(-\tfrac{53}{16}\theta_1^2\beta^3 + \tfrac{21}{16}\beta^5)\cos\tau]\mu^5 + \cdots\}\Big|.
\end{aligned}
$$

§ 13. *Non-homogeneous linear differential equations with periodic coefficients.*

If we have a set of homogeneous linear differential equations

$$
(84)\qquad \frac{dx_i}{dt} = \sum_{j=1}^{n}\theta_{ij}(t)x_j, \qquad (i = 1, \cdots, n),
$$

where the θ_{ij} are periodic functions of t with the period 2π, we know from the writings of Floquet * that, if the roots of the fundamental equation are all distinct, the solution has the form

$$
(85)\qquad x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t),
$$

where the A_j are arbitrary constants, the α_j are constants known as the characteristic exponents, and the ϕ_{ij} are periodic functions of t with the period 2π.

Such equations arise in dynamics whenever we study small variations from a known periodic solution. If we confine our attention entirely to the first powers of the variations the equations are linear and homogeneous, and are known as the "equations of variation." They have been studied extensively by Poincaré in *Les Méthodes Nouvelles de la Mécanique Céleste.* If the second and higher

* Annales Scientifiques de l'Ecole Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

powers of the variation are considered equations of the same type arise but they are no longer homogeneous. It is necessary for our purpose to derive the character of the solutions of such equations when the non-homogeneous terms are periodic, even though the period be different from 2π. We will suppose that the equations are the same as (84) with the addition of periodic terms.

Case I.

We will assume first that the period of the non-homogenous terms is 2π. The equations are then

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + g_i(t) \qquad (i=1, \cdots, n), \tag{86}$$

where $\theta_{ij}(t)$ and $g_i(t)$ are periodic with the period 2π. The solutions of (86) may be written

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t} \phi_{ij}(t) + \psi_i(t) \qquad (i=1, \cdots, n). \tag{87}$$

If in the differential equations we change t into $t + 2\pi$ the equations remain unchanged, but the solutions become

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j(t+2\pi)} \phi_{ij}(t + 2\pi) + \psi_i(t + 2\pi). \tag{88}$$

Therefore equations (88) are also solutions of (86), and consequently

$$\frac{d}{dt}\psi_i(t + 2\pi) = \sum_{j=1}^{n} \theta_{ij}\psi_j(t + 2\pi) + g_i(t);$$

also

$$\frac{d}{dt}\psi_i(t) = \sum_{j=1}^{n} \theta_{ij}\psi_j(t) + g_i(t).$$

Forming the difference of these equations we find

$$\frac{d}{dt}[\psi_i(t + 2\pi) - \psi_i(t)] = \sum_{j=1}^{n} \theta_{ij}[\psi_j(t + 2\pi) - \psi_j(t)] \quad (i=1, \cdots, n). \tag{89}$$

These equations are the same as (84) and their solutions have the same form as (85), that is

$$\psi_i(t + 2\pi) - \psi_i(t) = \sum_{j=1}^{n} B_j e^{\alpha_j t} \phi_{ij}(t). \tag{90}$$

The constants B_j in this case are not arbitrary but depend upon the differential equations. They may or may not be zero. Equations (90) may be interpreted as showing that $\psi_i(t)$ is composed of two parts, a periodic part and a non-

periodic part. We may express it therefore in the form

$$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j e^{\alpha_j t} \phi_{ij}(t) f_{ij}(t), \tag{91}$$

where $\omega_i(t)$ are periodic with the period 2π, and the $f_{ij}(t)$ are functions which must be determined. Changing t into $t + 2\pi$ in equations (91) and forming the difference $\psi_i(t + 2\pi) - \psi_i(t)$, we find

$$\psi_i(t + 2\pi) - \psi_i(t) = \sum_{j=1}^{n} B_j e^{\alpha_j t} \phi_{ij}(t)[e^{2\alpha_j \pi} f_{ij}(t + 2\pi) - f_{ij}(t)], \tag{92}$$

$$= \sum_{j=1}^{n} B_j e^{\alpha_j t} \phi_{ij}(t) \quad \text{from equations (90).}$$

Comparing the coefficients in these equalities we see that

$$e^{2\alpha_j \pi} f_{ij}(t + 2\pi) - f_{ij}(t) = 1, \tag{93}$$

from which we can determine the character of the functions $f_{ij}(t)$.

For this purpose let us define a new function $\lambda_{ij}(t)$ such that

$$\lambda_{ij}(t) = e^{\alpha_j t} f_{ij}(t) + \frac{e^{\alpha_j t}}{1 - e^{2\alpha_j \pi}}. \tag{94}$$

Then by virtue of the relations (93) the $\lambda_{ij}(t)$ are periodic with the period 2π, for we have

$$\lambda_{ij}(t + 2\pi) = e^{\alpha_j t} \cdot e^{2\alpha_j \pi} f_{ij}(t + 2\pi) + \frac{e^{\alpha_j t} \cdot e^{2\alpha_j \pi}}{1 - e^{2\alpha_j \pi}},$$

$$= e^{\alpha_j t}[f_{ij}(t) + 1] + \frac{e^{\alpha_j t} \cdot e^{2\alpha_j \pi}}{1 - e^{2\alpha_j \pi}},$$

$$= e^{\alpha_j t} f_{ij}(t) + \frac{e^{\alpha_j t}}{1 - e^{2\alpha_j \pi}},$$

$$= \lambda_{ij}(t).$$

On solving (94) we find

$$f_{ij}(t) = e^{-\alpha_j t} \lambda_{ij}(t) + \frac{1}{e^{2\alpha_j \pi} - 1}, \tag{95}$$

where the λ_{ij} are periodic with the period 2π. This expression for the $f_{ij}(t)$ substituted in (91) gives

$$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j \phi_{ij} \lambda_{ij} + \sum_{j=1}^{n} \frac{B_j}{e^{2\alpha_j \pi} - 1} e^{\alpha_j t} \phi_{ij}(t). \tag{96}$$

The terms included under the last summation sign are merely terms of the complementary function. All the other terms are periodic with the period 2π.

This form for the $\psi_i(t)$ fails however if for any j

$$e^{2\alpha_j\pi} = 1,$$

that is, if

$$\alpha_j \equiv 0 \mod \sqrt{-1}.$$

In this event equations (93) for such values of j become

$$f_{ij}(t + 2\pi) - f_{ij}(t) = 1. \tag{97}$$

We will define the corresponding value of λ_{ij} by the relation

$$\lambda_{ij}(t) = f_{ij}(t) - \frac{t}{2\pi}. \tag{98}$$

By virtue of (97) λ_{ij} is periodic with the period 2π, for

$$\begin{aligned}\lambda_{ij}(t + 2\pi) &= f_{ij}(t + 2\pi) - \frac{t + 2\pi}{2\pi},\\ &= f_{ij}(t) - \frac{t}{2\pi},\\ &= \lambda_{ij}(t).\end{aligned}$$

Solving (98) for $f_{ij}(t)$, we obtain

$$f_{ij}(t) = \lambda_{ij}(t) + \frac{t}{2\pi}. \tag{99}$$

Substituting this expression in (91), we find

$$\psi_i(t) = \text{periodic terms} + \frac{t}{2\pi}\sum_j B_j e^{\alpha_j t}\phi_{ij}(t), \tag{100}$$

the summation in the last term to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

We have then the general expression for the solution

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t) + \text{periodic terms} + \frac{t}{2\pi}\sum_j B_j e^{\alpha_j t}\phi_{ij}(t), \tag{101}$$

where the summation in the last term is to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

This result may be stated thus:

THEOREM I. *If the $\theta_{ij}(t)$ and $g_i(t)$ are periodic with the period 2π and if the characteristic exponents α_j are distinct and none of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution is periodic with the period 2π. If the α_j are distinct but some of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution may involve terms of the form t times the corresponding complementary function.*

Case II.

We will suppose that the $g_i(t)$ are periodic and that the period is different from 2π. We will suppose further that $g_i(t)$ are expressible in the form

$$(102)\quad g_i(t)=\sum_j [a_{ij}\cos(j+\beta)t+b_{ij}\sin(j+\beta)t]=e^{\sqrt{-1}\beta t}f_i^{(1)}(t)+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t),$$

where $f_i^{(1)}(t)$ and $f_i^{(2)}(t)$ are periodic with the period 2π and β is any real number not an integer. The differential equations may then be written

$$(103)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{\sqrt{-1}\beta t}f_i^{(1)}(t)+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t).$$

Since the equations are linear we can consider the particular solutions depending upon $e^{\sqrt{-1}\beta t}$ and $e^{-\sqrt{-1}\beta t}$ separately. The complete solution will be the sum of the two. Let us consider first

$$(104)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{\sqrt{-1}\beta t}f_i^{(1)}(t).$$

If we make the substitution

$$x_i=e^{\sqrt{-1}\beta t}y_i$$

equations (104) become after dividing out the exponential

$$(105)\qquad \frac{dy_i}{dt}+\sqrt{-1}\,y_i=\sum_{j=1}^{n}\theta_{ij}y_j+f_i^{(1)}(t).$$

Since θ_{ij} and $f_i^{(1)}$ are periodic with the period 2π the discussion of the character of the y_i reduces to Case I. The characteristic exponents of equations (105) are $(\alpha_j-\sqrt{-1}\beta)$, where the α_j are the characteristic exponents for the homogeneous equations (84). Applying Theorem I we conclude that the y_i are periodic with the period 2π provided

$$\alpha_j\not\equiv\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1,\cdots,n).$$

If, however, for any value of j

$$\alpha_j\equiv\sqrt{-1}\beta \mod \sqrt{-1}$$

then, in general, non-periodic terms will arise just as in Case 1.

The discussion for the second part of the differential equations

$$(106)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t)$$

shows similarly that the particular integral is periodic if

$$\alpha_j\not\equiv-\sqrt{-1}\beta \qquad (j=1,\cdots,n).$$

In equations arising in dynamics the characteristic exponents α_j enter in pairs, equal but of opposite sign. Consequently for such equations this last condition

$$\alpha_j \not\equiv -\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1,\cdots,n),$$

does not differ from

$$\alpha_j \not\equiv +\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1,\cdots,n).$$

We have then the following

THEOREM II. *If the θ_{ij} are periodic with the period 2π, and if the $g_i(t)$ have the form*

$$g_i(t) = \sum_k [a_{ik}\cos(k+\beta)t + b_{ik}\sin(k+\beta)t],$$

and if the characteristic exponents α_j are distinct and none of them congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution has the form

$$x_i = \sum_m [c_{im}\cos(m+\beta)t + d_{im}\sin(m+\beta)t].$$

If any of the α_j are congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution will, in general, contain, in addition to periodic terms, terms of the form t times the corresponding complementary function.

Let us suppose that one of the α_j is congruent to $\sqrt{-1}\beta \mod \sqrt{-1}$. Then, in general, the solution will contain non-periodic terms. If now the $g_i(t)$ contain also terms of the form $\frac{\sin}{\cos}(p+q\beta)t$, where p and q are integers, these terms will not, in general, give rise to non-periodic terms in the solution, for the α_j will not, in general, be congruent to $\sqrt{-1}\,q\beta$. Indeed, if β is not rational, for no integral value of q except unity will $\sqrt{-1}\,q\beta \equiv \alpha_j$. But if β (and therefore α_j also) is rational, then certain values of q do exist for which $\alpha_j \equiv \sqrt{-1}\,q\beta$. Let us set

$$\alpha = \frac{i}{j}\sqrt{-1}, \qquad \beta = \frac{I}{J},$$

where i and j are integers relatively prime, and similarly for I and J. By hypothesis

$$\frac{i}{j} - \frac{I}{J} \equiv 0 \mod 1;$$

then

$$\frac{i}{j} - q\frac{I}{J} \not\equiv 0 \mod 1$$

unless $q = 1 + lJ$ (l an integer). In this event a term of the form

$\begin{matrix}\cos\\ \sin\end{matrix}(p+q\beta)t$ can also be written $\begin{matrix}\cos\\ \sin\end{matrix}(r+\beta)t$ (r an integer) which involves only the first multiple of β, and the solution in general involves non-periodic terms.

Case III.

When two of the characteristic exponents of the homogeneous set are equal the preceding arguments are not applicable. Let us suppose that the homogeneous equations are

$$\frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j \qquad (i=1,\cdots,n), \tag{107}$$

but that $\alpha_{n-1}=\alpha_n$. The solution then has in general the form

$$x_i=\sum_{j=1}^{n-2}C_j e^{\alpha_j t}\phi_{ij}(t)+[A+Bt]e^{\alpha_n t}\phi_{in}(t)+Be^{\alpha_n t}\phi_{i(n-1)}(t).$$

Suppose now the given equations are

$$\frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+g_i(t),$$

where the θ_{ij} are the same as in (107) and the g_i are periodic functions of t with the period 2π. Let the particular solutions which depend on the $g_i(t)$ be

$$x_i=\psi_i(t).$$

Just as in Case I we find

$$\frac{d}{dt}[\psi_i(t+2\pi)-\psi_i(t)]=\sum_{j=1}^{n}\theta_{ij}[\psi_j(t+2\pi)-\psi_j(t)], \tag{108}$$

and consequently

$$\psi_i(t+2\pi)-\psi_i(t)=\sum_{j=1}^{n-2}C'_j e^{\alpha_j t}\phi_{ij}(t)+[A'+B't]e^{\alpha_n t}\phi_{in}(t)+B'e^{\alpha_n t}\phi_{i(n-1)}(t). \tag{109}$$

The terms included under the summation sign are obviously the same as those treated in Case I. Neglecting these we may write

$$\psi_i(t)=e^{\alpha_n t}\phi_{in}(t)\cdot f_{in}(t)+e^{\alpha_n t}\cdot\phi_{i(n-1)}(t)f_{i(n-1)}(t)+\text{periodic terms}, \tag{110}$$

where $f_{in}(t)$ and $f_{i(n-1)}(t)$ are functions whose forms are to be determined. Let us form the difference $\psi_i(t+2\pi)-\psi_i(t)$ by means of (110) and compare the result with (109). From (110) we get

$$\begin{aligned}\psi_i(t+2\pi)-\psi_i(t)=&[e^{2\alpha_n\pi}f_{in}(t+2\pi)-f_{in}(t)]\,e^{\alpha_n t}\phi_{in}(t)\\&+[e^{2\alpha_n\pi}f_{i(n-1)}(t)-f_{i(n-1)}(t)]\,e^{\alpha_n t}\phi_{i(n-1)}(t).\end{aligned} \tag{111}$$

Comparing this with (109) we see that we must have

$$(112)\qquad \begin{aligned} e^{2a_n\pi}f_{in}(t+2\pi)-f_{in}(t)&=A'+B't,\\ e^{2a_n\pi}f_{i(n-1)}(t+2\pi)-f_{i(n-1)}(t)&=B'. \end{aligned}$$

Let us now define two new functions

$$(113)\qquad \begin{aligned} \lambda_{in}(t)&=e^{a_nt}f_{in}(t)+\left[\frac{A'}{1-e^{2a_n\pi}}+\frac{2\pi B'e^{2a_n\pi}}{(1-e^{2a_n\pi})^2}\right]e^{a_nt}+\frac{B'te^{a_nt}}{1-e^{2a_n\pi}},\\ \lambda_{i(n-1)}(t)&=e^{a_nt}f_{i(n-1)}(t)+\frac{B'e^{a_nt}}{1-e^{2a_n\pi}}. \end{aligned}$$

By virtue of the relations (112), $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π, and consequently

$$(114)\qquad \begin{aligned} f_{in}(t)&=e^{-a_nt}\lambda_{in}(t)+\left[\frac{A'}{e^{2a_n\pi}-1}-\frac{2\pi B'e^{2a_n\pi}}{(e^{2a_n\pi}-1)^2}\right]+\frac{B't}{e^{2a_n\pi}-1},\\ f_{i(n-1)}(t)&=e^{-a_nt}\lambda_{i(n-1)}+\frac{B'}{e^{2a_n\pi}-1}, \end{aligned}$$

where $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π. Substituting these expressions in (110), we find

$$(115)\qquad \begin{aligned} \psi_i(t)=&\left[\frac{A'}{(e^{2a_n\pi}-1)}-\frac{2\pi B'e^{2a_n\pi}}{(e^{2a_n\pi}-1)^2}+\frac{B't}{(e^{2a_n\pi}-1)}\right]e^{a_nt}\phi_{in}(t)\\ &+\frac{B'}{(e^{2a_n\pi}-1)}e^{a_nt}\phi_{i(n-1)}(t)+\text{periodic terms}. \end{aligned}$$

Comparing these terms with (109) we see that they are merely terms of the complementary function and that therefore

$$x_i=\text{complementary function}+\text{periodic terms}.$$

These results hold provided $\alpha_n\not\equiv 0 \bmod \sqrt{-1}$. In the event $\alpha_n\equiv 0 \bmod \sqrt{-1}$ equations (112) become

$$(116)\qquad f_{in}(t+2\pi)-f_{in}(t)=A'+B't,\qquad f_{i(n-1)}(t+2\pi)-f_{i(n-1)}(t)=B'.$$

It is necessary to give new definitions to the λ functions. We will let

$$(117)\qquad \lambda_{in}(t)=f_{in}(t)-\frac{B'}{4\pi}t^2-\frac{A'-B'\pi}{2\pi}t,\qquad \lambda_{i(n-1)}(t)=f_{i(n-1)}(t)-\frac{B'}{2\pi}t,$$

which give

$$(118)\qquad f_{in}(t)=\lambda_{in}(t)+\frac{B'}{4\pi}t^2+\frac{A'-B'\pi}{2\pi}t,\qquad f_{i(n-1)}(t)=\lambda_{i(n-1)}(t)+\frac{B'}{2\pi}t.$$

It is readily verified that $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic by virtue of (116). Substituting these expressions in (110), we find

$$(119)\quad \psi_i(t)=e^{\alpha_n t}\left\{\left[\frac{B'}{4\pi}t^2+\frac{A'-B'\pi}{2\pi}t\right]\phi_{in}(t)+\frac{B'}{2\pi}t\phi_{i(n-1)}(t)\right\}+\text{periodic terms.}$$

If we let

$$\frac{A'-B'\pi}{2\pi}=C,\qquad \frac{B'}{2\pi}=D,$$

we get

$$\psi_i(t)=e^{\alpha_n t}\{[Ct+\tfrac{1}{2}Dt^2]\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}+\text{periodic terms.}$$

This expression is not quite the same as t times the corresponding complementary function for in one term we have the coefficient $\frac{1}{2}D$ instead of D. The theorem for this case then is

THEOREM III. *If θ_{ij} and $g_i(t)$ are periodic with the period 2π, and if two of the α_j are equal but not congruent to zero* mod $\sqrt{-1}$, *then the particular solution consists of terms periodic with the period 2π plus a constant times the corresponding complementary function. But if two of the α_j are equal and congruent to zero so that the corresponding part of the complementary function has the form*

$$e^{\alpha_n t}\{(A+Bt)\phi_{in}(t)+B\phi_{i(n-1)}(t)\},$$

then the particular solution consists of a periodic function plus a term of the form

$$e^{\alpha_n t}\{(Ct+\tfrac{1}{2}Dt^2)\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}.$$

§ 14. *Special theorems for the equations of variation.*

The foregoing theorems presuppose merely the conditions that the coefficients are periodic with the period 2π. Further facts with regard to the solutions may be established when further facts are specified with regard to the coefficients of the differential equations. Our equations of variation (51) may be written

$$(120)\quad \frac{d\rho_1}{d\tau}=\rho_2,\qquad \frac{d\rho_2}{d\tau}=\bar{\theta}_2\rho_1+\theta_3\sigma_1,\qquad \frac{d\sigma_1}{d\tau}=\sigma_2,\qquad \frac{d\sigma_2}{d\tau}=\theta_3\rho_1+\bar{\theta}_4\sigma_1,$$

where the notation *with respect to the* θ's has the following significance: even subscripts denote functions even in τ, and odd subscripts denote functions odd in τ; one dash indicates that only odd multiples of τ are involved, and two dashes indicate that only even multiples of τ are involved. The solutions, equations (82) and (83), may be characterized in the same manner, and are

then

$$
\begin{aligned}
\rho_1 &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\bar{\alpha}_3(\tau) + D[\bar{\alpha}_4(\tau) + \tau\bar{\alpha}_3(\tau)],\\
\rho_2 &= A\beta_1(\tau) + B\beta_2(\tau) + C\bar{\beta}_4(\tau) + D[\bar{\beta}_3(\tau) + \tau\bar{\beta}_4(\tau)],\\
\sigma_1 &= A\bar{\gamma}_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)],\\
\sigma_2 &= A\bar{\delta}_2(\tau) + B\bar{\delta}_1(\tau) + C\delta_3(\tau) + D[\delta_4(\tau) + \tau\delta_3(\tau)],
\end{aligned}
\tag{121}
$$

where the notation is the same as for the θ's with the exception that in the first two solutions every integral multiple of τ is increased by $\pm\lambda\tau$, e. g., $\cos(3+\lambda)\tau$. On these terms the dashes refer only to the integral part of the coefficients of τ.

Suppose now we have the following non-homogeneous equations:

$$
\begin{aligned}
\frac{d\rho_1}{d\tau} &= \rho_2, \qquad \frac{d\rho_2}{d\tau} = \bar{\theta}_2\rho_1 + \theta_3\sigma_1 + g(\tau),\\
\frac{d\sigma_1}{d\tau} &= \sigma_2, \qquad \frac{d\sigma_2}{d\tau} = \theta_3\rho_1 + \bar{\theta}_4\sigma_1 + f(\tau),
\end{aligned}
\tag{122}
$$

where $g(\tau)$ and $f(\tau)$ are periodic with the period 2π. Since the characteristic exponents are $\sqrt{-1}\lambda$, $-\sqrt{-1}\lambda$, 0, 0, by Theorem III the solution has the form

$$
\begin{aligned}
\rho_1 &= (\rho_1) + \xi_1 = (\rho_1) + \omega_1(\tau) + a\tau\bar{\alpha}_3 + b[\tfrac{1}{2}\tau^2\bar{\alpha}_3 + \tau\bar{\alpha}_4],\\
\rho_2 &= (\rho_2) + \xi_2 = (\rho_2) + \omega_2(\tau) + a\tau\bar{\beta}_4 + b[\tfrac{1}{2}\tau^2\bar{\beta}_4 + \tau\bar{\beta}_3],\\
\sigma_1 &= (\sigma_1) + \eta_1 = (\sigma_1) + \omega_3(\tau) + a\tau\gamma_4 + b[\tfrac{1}{2}\tau^2\gamma_4 + \tau\gamma_3],\\
\sigma_2 &= (\sigma_1) + \eta_2 = (\sigma_2) + \omega_4(\tau) + a\tau\delta_3 + b[\tfrac{1}{2}\tau^2\delta_3 + \tau\delta_4],
\end{aligned}
\tag{123}
$$

where (ρ_i), (σ_i) indicate the complementary function and ξ_i and η_i are the particular integrals, of which the ω_i are the periodic parts. The a and b are constants which depend upon the differential equations. Since the ξ_i and η_i satisfy the differential equations we have

$$
\begin{aligned}
\frac{d}{d\tau}\xi_1(\tau) &= \xi_2(\tau), \qquad \frac{d}{d\tau}\xi_2(\tau) = \bar{\theta}_2\xi_1(\tau) + \theta_3\eta_1(\tau) + g(\tau),\\
\frac{d}{d\tau}\eta_1(\tau) &= \eta_2(\tau), \qquad \frac{d}{d\tau}\eta_2(\tau) = \theta_3\xi_1(\tau) + \bar{\theta}_4\eta_1(\tau) + f(\tau).
\end{aligned}
\tag{124}
$$

Changing τ into $-\tau$ in these equations, we get

$$
\begin{aligned}
\frac{d}{d\tau}\xi_1(-\tau) &= -\xi_2(-\tau), \qquad \frac{d}{d\tau}\xi_2(-\tau) = -\bar{\theta}_2\xi_1(-\tau) + \theta_3\eta_1(-\tau) - g(-\tau),\\
\frac{d}{d\tau}\eta_1(-\tau) &= -\eta_2(-\tau), \qquad \frac{d}{d\tau}\eta_2(-\tau) = +\theta_3\xi_1(-\tau) - \bar{\theta}_4\eta_1(-\tau) - f(-\tau).
\end{aligned}
\tag{125}
$$

If now we make the additional hypothesis that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, equations (124) and (125) may be combined into the following set

$$
\begin{aligned}
&\frac{d}{d\tau}[\xi_1(\tau)-\xi_1(-\tau)]=[\xi_2(\tau)+\xi_2(-\tau)],\\
&\frac{d}{d\tau}[\xi_2(\tau)+\xi_2(-\tau)]=\theta_2[\xi_1(\tau)-\xi_1(-\tau)]+\theta_3[\eta_1(\tau)+\eta_1(-\tau)],\\
&\frac{d}{d\tau}[\eta_1(\tau)+\eta_1(-\tau)]=[\eta_2(\tau)-\eta_2(-\tau)],\\
&\frac{d}{d\tau}[\eta_2(\tau)-\eta_2(-\tau)]=\theta_3[\xi_1(\tau)-\xi_1(-\tau)]+\theta_4[\eta_1(\tau)+\eta_1(-\tau)].
\end{aligned}
\tag{126}
$$

These equations are the same as (120). Therefore

$$
\begin{aligned}
&\xi_1(\tau)-\xi_1(-\tau)=A\alpha_2(\tau)+B\alpha_1(\tau)+C\alpha_3(\tau)+D[\alpha_4(\tau)+\tau\alpha_3(\tau)],\\
&\xi_2(\tau)+\xi_2(-\tau)=A\beta_1(\tau)+B\beta_2(\tau)+C\beta_4(\tau)+D[\beta_3(\tau)+\tau\beta_4(\tau)],\\
&\eta_1(\tau)+\eta_1(-\tau)=A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\gamma_4(\tau)],\\
&\eta_2(\tau)-\eta_2(-\tau)=A\delta_2(\tau)+B\delta_1(\tau)+C\delta_3(\tau)+D[\delta_4(\tau)+\tau\delta_3(\tau)].
\end{aligned}
\tag{127}
$$

Putting $\tau=0$ in these equations we get from the first and the fourth

$$
0=A\alpha_2(0)+D\alpha_4(0), \qquad 0=A\delta_2(0)+D\delta_4(0). \tag{128}
$$

Either $A=D=0$, or the determinant $\alpha_2(0)\delta_4(0)-\delta_2(0)\alpha_4(0)=0$. But it is readily verified that the determinant is not zero. Therefore $A=D=0$. If we suppose that $\xi_2(0)=\eta_1(0)=0$ (we shall be interested only in such cases), it follows from the second and third equations of (127) that

$$
0=B\beta_2(0)+C\beta_4(0), \qquad 0=B\gamma_2(0)+C\gamma_4(0),
$$

and hence $B=C=0$. Consequently

$$
\begin{aligned}
&\xi_1(\tau)-\xi_1(-\tau)=0, \qquad \xi_2(\tau)+\xi_2(-\tau)=0,\\
&\eta_1(\tau)+\eta_1(-\tau)=0, \qquad \eta_2(\tau)-\eta_2(-\tau)=0.
\end{aligned}
\tag{129}
$$

We have then

THEOREM IV. *If $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function of τ, and if $\xi_2(0)=\eta_1(0)=0$, then $\xi_1(\tau)$ and $\eta_2(\tau)$ are even functions of τ, and $\xi_2(\tau)$ and $\eta_1(\tau)$ are odd functions of τ.*

In the same way it can be shown that if $g(\tau)$ is odd and $f(\tau)$ is even, and if $\xi_1(0) = \eta_2(0) = 0$, then ξ_1 and η_2 are odd and ξ_2 and η_1 are even.

Let us suppose now that $g(\tau)$ is periodic and contains only even multiples of τ, and that $f(\tau)$ is periodic and contains only odd multiples of τ. The general form of the solution will be the same as (123). ξ_1, ξ_2, η_1, and η_2 satisfy the differential equations

$$(130)\quad \begin{aligned} &\frac{d}{d\tau}\xi_1(\tau) = \xi_2(\tau), \qquad &&\frac{d}{d\tau}\xi_2(\tau) = \bar{\theta}_2\xi_1(\tau) + \theta_3\eta_1(\tau) + \bar{g}(\tau),\\ &\frac{d}{d\tau}\eta_1(\tau) = \eta_2(\tau), \qquad &&\frac{d}{d\tau}\eta_2(\tau) = \theta_3\xi_1(\tau) + \bar{\theta}_4\eta_1(\tau) + f(\tau). \end{aligned}$$

Let us denote $\xi_i(\tau + \pi)$ by $\xi_i'(\tau)$ and $\eta_i(\tau + \pi)$ by $\eta_i'(\tau)$. Then by changing τ into $\tau + \pi$ in (130) we have

$$(131)\quad \begin{aligned} &\frac{d}{d\tau}\xi_1'(\tau) = \xi_2'(\tau), \qquad &&\frac{d}{d\tau}\xi_2'(\tau) = \bar{\theta}_2\xi_1'(\tau) - \theta_3\eta_1'(\tau) + \bar{g}(\tau),\\ &\frac{d}{d\tau}\eta_1'(\tau) = \eta_2'(\tau), \qquad &&\frac{d}{d\tau}\eta_2'(\tau) = -\theta_3\xi_1'(\tau) + \bar{\theta}_4\eta_1'(\tau) - f(\tau). \end{aligned}$$

From (130) and (131) it follows that

$$(132)\quad \begin{aligned} &\frac{d}{d\tau}[\xi_1 - \xi_1'] = [\xi_2 - \xi_2'], \qquad &&\frac{d}{d\tau}[\xi_2 - \xi_2'] = \bar{\theta}_2[\xi_1 - \xi_1'] + \theta_3[\eta_1 + \eta_1'],\\ &\frac{d}{d\tau}[\eta_1 + \eta_1'] = [\eta_2 + \eta_2'], \qquad &&\frac{d}{d\tau}[\eta_2 + \eta_2'] = \theta_3[\xi_1 - \xi_1'] + \bar{\theta}_4[\eta_1 + \eta_1']. \end{aligned}$$

The solutions of these equations, which have the form of (120), are

$$(133)\quad \begin{aligned} \xi_1 - \xi_1' &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\bar{\alpha}_3(\tau) + D[\bar{\alpha}_4(\tau) + \tau\bar{\alpha}_3(\tau)],\\ \xi_2 - \xi_2' &= A\beta_1(\tau) + B\beta_2(\tau) + C\bar{\beta}_4(\tau) + D[\bar{\beta}_3(\tau) + \tau\bar{\beta}_4(\tau)],\\ \eta_1 + \eta_1' &= A\bar{\gamma}_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)],\\ \eta_2 + \eta_2' &= A\bar{\delta}_2(\tau) + B\bar{\delta}_1(\tau) + C\delta_3(\tau) + D[\delta_4(\tau) + \tau\delta_3(\tau)]. \end{aligned}$$

Forming these expressions directly from (123), we get

$$(134)\quad \begin{aligned} \xi_1 - \xi_1' &= \omega_1(\tau) - \omega_1(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\alpha}_3(\tau) - b\pi[\tau\bar{\alpha}_3(\tau) + \bar{\alpha}_4(\tau)],\\ \xi_2 - \xi_2' &= \omega_2(\tau) - \omega_2(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\beta}_4(\tau) - b\pi[\tau\bar{\beta}_4(\tau) + \bar{\beta}_3(\tau)],\\ \eta_1 + \eta_1' &= \omega_3(\tau) + \omega_3(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\gamma}_4(\tau) - b\pi[\tau\gamma_4(\tau) + \gamma_3(\tau)],\\ \eta_2 + \eta_2' &= \omega_4(\tau) + \omega_4(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\delta_3(\tau) - b\pi[\tau\delta_3(\tau) + \delta_4(\tau)]. \end{aligned}$$

Comparing (133) and (134) we see that

$$A = B = 0, \qquad C = -[a\pi + \tfrac{1}{2}b\pi^2], \qquad D = -b\pi,$$
$$\omega_1(\tau) - \omega_1(\tau + \pi) = 0, \qquad \omega_3(\tau) + \omega_3(\tau + \pi) = 0,$$
$$\omega_2(\tau) - \omega_2(\tau + \pi) = 0, \qquad \omega_4(\tau) + \omega_4(\tau + \pi) = 0.$$

Therefore $\omega_1(\tau)$ and $\omega_2(\tau)$ contain only even multiples of τ while $\omega_3(\tau)$ and $\omega_4(\tau)$ contain only odd multiples, and by carrying this result into (123) we find

$$\begin{aligned}
\xi_1 &= \bar{\omega}_1(\tau) + a\tau\bar{\alpha}_3(\tau) + b\,[\tfrac{1}{2}\tau^2\bar{\alpha}_3(\tau) + \tau\bar{\alpha}_4(\tau)],\\
\xi_2 &= \bar{\omega}_2(\tau) + a\tau\bar{\beta}_4(\tau) + b\,[\tfrac{1}{2}\tau^2\bar{\beta}_4(\tau) + \tau\bar{\beta}_3(\tau)],\\
\eta_1 &= \bar{\omega}_3(\tau) + a\tau\gamma_4(\tau) + b\,[\tfrac{1}{2}\tau^2\gamma_4(\tau) + \tau\gamma_3(\tau)],\\
\eta_2 &= \bar{\omega}_4(\tau) + a\tau\delta_3(\tau) + b\,[\tfrac{1}{2}\tau^2\delta_3(\tau) + \tau\delta_4(\tau)],
\end{aligned} \tag{135}$$

and hence we have

THEOREM V. *If $g(\tau)$ contains only even multiples of τ and $f(\tau)$ contains only odd multiples, then ξ_1 and ξ_2 contain only even multiples of τ and η_1 and η_2 contain only odd multiples.*

If in addition to the above hypotheses we suppose that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, then ξ_1 and η_2 are even functions of τ and ξ_2 and η_1 are odd functions. Therefore $b = 0$. But if $g(\tau)$ is an odd function and $f(\tau)$ is an even function, then ξ_1 and η_2 are odd functions and ξ_2 and η_1 are even functions, so that in this event $a = 0$.

In the same manner as above we prove

THEOREM VI. *If $g(\tau)$ contains only odd multiples of τ and $f(\tau)$ contains only even multiples, then ξ_1 and ξ_2 contain only odd multiples of τ and η_1 and η_2 contain only even multiples. Furthermore ξ_1, ξ_2, η_1 and η_2 are periodic with the period 2π.*

If $g(\tau)$ is of the form $\sum_j m_j \cos(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \sin(j \pm \lambda)\tau$ then, since $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, the form of the solution is, by Theorem II,

$$\begin{aligned}
\xi_1 &= \sum_k p_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k p_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\alpha_1(\tau) + a^{(2)}\tau\bar{\alpha}_2(\tau),\\
\xi_2 &= \sum_k q_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k q_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\beta_2(\tau) + a^{(2)}\tau\beta_1(\tau),\\
\eta_1 &= \sum_k r_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k r_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\gamma}_2(\tau) + a^{(2)}\tau\bar{\gamma}_1(\tau),\\
\eta_2 &= \sum_k s_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k s_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\delta}_1(\tau) + a^{(2)}\tau\bar{\delta}_2(\tau);
\end{aligned} \tag{136}$$

but since $g(\tau)$ is an even function and $f(\tau)$ is an odd function, ξ_1 and η_2 are

even functions and ξ_2 and η_1 are odd functions. Therefore all the coefficients in (136) which have the upper index (2) are zero. But if $g(\tau)$ were an odd function of τ and $f(\tau)$ an even function then all the coefficients in (136) which have the upper index (1) would be zero. Therefore

THEOREM VII. *If $g(\tau)$ is of the form $\sum_j m_j \cos(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \sin(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a p_a \cos(a \pm \lambda)\tau + A\tau\alpha_1(\tau),$$

$$\xi_2 = \sum_b p_b \sin(b \pm \lambda)\tau + A\tau\beta_2(\tau),$$

$$\eta_1 = \sum_c p_c \sin(c \pm \lambda)\tau + A\tau\bar{\gamma}_2(\tau),$$

$$\eta_2 = \sum_d p_d \cos(d \pm \lambda)\tau + A\tau\bar{\delta}_1(\tau).$$

Also,

THEOREM VIII. *If $g(\tau)$ has the form $\sum_j m_j \sin(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \cos(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a r_a \sin(a \pm \lambda)\tau + B\tau\alpha_2(\tau), \qquad \xi_2 = \sum_b r_b \cos(b \pm \lambda)\tau + B\tau\beta_1(\tau),$$

$$\eta_1 = \sum_c r_c \cos(c \pm \lambda)\tau + B\tau\bar{\gamma}_1(\tau), \qquad \eta_2 = \sum_d r_d \sin(d \pm \lambda)\tau + B\tau\bar{\delta}_2(\tau).$$

It is understood that in the above two theorems a, b, c, d, and j are integers.

§ 15. *Integration of the complete differential equations* (50).

It will be convenient hereafter to use the following notation for the solutions of the equations of variation:

$$\begin{aligned} \rho &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)], \\ \sigma &= A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_1(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)]. \end{aligned} \tag{137}$$

The $\alpha_j(\tau)$ and $\gamma_j(\tau)$ are characterized thus:

$\alpha_2(\tau)$	involves	only	terms	of	the	form	$\cos[(2n+1) \pm \lambda]\tau$,
$\gamma_1(\tau)$	"	"	"	"	"	"	$\sin[2n \pm \lambda]\tau$,
$\alpha_1(\tau)$	"	"	"	"	"	"	$\sin[(2n+1) \pm \lambda]\tau$,
$\gamma_2(\tau)$	"	"	"	"	"	"	$\cos[2n \pm \lambda]\tau$,
$\alpha_3(\tau)$	"	"	"	"	"	"	$\sin[2n]\tau$,
$\gamma_4(\tau)$	"	"	"	"	"	"	$\cos[2n+1]\tau$,
$\alpha_4(\tau)$	"	"	"	"	"	"	$\cos[2n]\tau$,
$\gamma_3(\tau)$	"	"	"	"	"	"	$\sin[2n+1]\tau$.

It will be convenient also to write the differential equations for ρ and σ as

$$(138)\quad \begin{aligned} \rho'' + \theta_2\rho + \theta_3\sigma &= \theta_{001}\epsilon + \theta_{101}\epsilon\rho + \theta_{200}\rho^2 + \theta_{110}\rho\sigma + \theta_{020}\sigma^2 + \cdots, \\ \sigma'' + \theta_4\sigma + \theta_3\rho &= \qquad\qquad \theta_{200}\rho^2 + \theta_{110}\rho\sigma + \theta_{020}\sigma^2 + \cdots, \end{aligned}$$

where all the θ's are periodic with the period 2π, and θ_2 and θ_4 contain only cosines of even multiples of τ, and θ_3 contains only sines of odd multiples of τ. On the right side of the first equation the coefficients of terms carrying odd powers of σ contain only sines of odd multiples of τ. All the other coefficients contain only cosines of even multiples. In the second equation odd powers of σ have coefficients involving only cosines of even multiples. All other coefficients contain only sines of odd multiples.

The initial conditions are

$$\rho = \alpha, \qquad \sigma = 0, \qquad \rho' = 0, \qquad \sigma' = \delta.$$

We will integrate equations (138) as power series in α, δ and ϵ, τ entering into the coefficients. From Poincaré's extension of Cauchy's theorem we know that these series are convergent for any arbitrarily chosen interval for τ, $0 \leqq \tau \leqq T$, provided $|\alpha|$, $|\delta|$ and $|\epsilon|$ are sufficiently small. The equations of variation involve the period $2\pi/\lambda$. The solutions are not periodic unless λ is rational. Hence θ_1^2 must be chosen in advance so that λ is rational. We will suppose then that $\lambda = J/K$, where J and K are integers, relatively prime. Then three solutions of the equations of variation are periodic with the period $2K\pi$.

Since ρ and σ are expansible in powers of α, δ and ϵ, we may write

$$\rho = \rho_{100}\alpha + \rho_{010}\delta + \rho_{001}\epsilon + \rho_{200}\alpha^2 + \rho_{110}\alpha\delta + \rho_{020}\delta^2 + \rho_{101}\alpha\epsilon + \rho_{011}\delta\epsilon + \rho_{002}\epsilon^2 + \cdots,$$

$$\sigma = \sigma_{100}\alpha + \sigma_{010}\delta + \sigma_{001}\epsilon + \sigma_{200}\alpha^2 + \sigma_{110}\alpha\delta + \sigma_{020}\delta^2 + \sigma_{101}\alpha\epsilon + \sigma_{011}\rho\epsilon + \sigma_{002}\epsilon^2 + \cdots.$$

Substituting these expressions in the differential equations (138) and equating the coefficients of similar powers, we have:

Coefficient of α.

$$\rho'_{100} + \theta_2\rho_{100} + \theta_3\sigma_{100} = 0,$$

$$\sigma''_{100} + \theta_4\sigma_{100} + \theta_3\rho_{100} = 0.$$

The solution of these equations is

$$\rho_{100} = A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)],$$

$$\sigma_{100} = A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)].$$

In order to satisfy the initial conditions we must have, at $\tau = 0$,

$$\rho_{100} = 1, \qquad \sigma_{100} = 0, \qquad \rho'_{100} = 0, \qquad \sigma'_{100} = 0.$$

From these conditions we find

$$
\begin{aligned}
&A\alpha_2(0) + D\alpha_4(0) = 1,\\
&B\alpha_1'(0) + C\alpha_3'(0) = 0,\\
&B\gamma_2(0) + C\gamma_4(0) = 0,\\
&A\gamma_1'(0) + D[\gamma_3'(0) + \gamma_4(0)] = 0.
\end{aligned}
\tag{139}
$$

The solution of these conditional equations is

$$
\begin{aligned}
&B = 0, \qquad A = \frac{\gamma_3'(0) + \gamma_4(0)}{\Delta} = A_{100}^{(1)},\\
&C = 0, \qquad D = -\frac{\gamma_1'(0)}{\Delta} = A_{100}^{(2)},\\
&\Delta = \alpha_2(0)[\gamma_3'(0) + \gamma_4(0)] - \alpha_4(0)\gamma_1'(0).
\end{aligned}
\tag{140}
$$

Hence the solution is

$$
\begin{aligned}
\rho_{100} &= A_{100}^{(1)}\alpha_2(\tau) + A_{100}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)],\\
\sigma_{100} &= A_{100}^{(1)}\gamma_1(\tau) + A_{100}^{(2)}[\gamma_3(\tau) + \tau\alpha_4(\tau)].
\end{aligned}
\tag{141}
$$

Coefficient of δ.

$$
\begin{aligned}
&\rho_{010}'' + \theta_2\rho_{010} + \theta_3\sigma_{010} = 0,\\
&\sigma_{010}'' + \theta_4\sigma_{010} + \theta_3\rho_{010} = 0.
\end{aligned}
\tag{142}
$$

These equations are the same as for the coefficient of α. From the initial conditions we must have at $\tau = 0$

$$
\rho_{010} = 0, \qquad \sigma_{010} = 0, \qquad \rho_{010}' = 0, \qquad \sigma_{010}' = 1.
$$

The solutions are

$$
\begin{aligned}
\rho_{010} &= A_{010}^{(1)}\alpha_2(\tau) + A_{010}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)],\\
\sigma_{010} &= A_{010}^{(1)}\gamma_1(\tau) + A_{010}^{(2)}[\gamma_3(\tau) + \tau\gamma_4(\tau)],
\end{aligned}
\tag{143}
$$

where

$$
A_{010}^{(1)} = -\frac{\alpha_4(0)}{\Delta}, \qquad A_{010}^{(2)} = \frac{\alpha_2(0)}{\Delta}.
$$

Coefficient of ϵ.

$$
\begin{aligned}
&\rho_{001}'' + \theta_2\rho_{001} + \theta_3\sigma_{001} = \theta_{001},\\
&\sigma_{001}'' + \theta_4\sigma_{001} + \theta_3\rho_{001} = 0.
\end{aligned}
\tag{144}
$$

The right member, θ_{001}, is a periodic function of τ with the period 2π. Furthermore it involves only cosines of even multiples of τ. Hence by Theorem V the

solution has the form

$$(145)\quad \begin{aligned}\rho_{001} &= A\alpha_2(\tau)+B\alpha_1(\tau)+C\alpha_3(\tau)+D[\alpha_4(\tau)+\tau\alpha_3(\tau)]+\alpha_5(\tau)+a\tau\alpha_3(\tau),\\ \sigma_{001} &= A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\gamma_4(\tau)]+\gamma_5(\tau)+a\tau\gamma_4(\tau),\end{aligned}$$

where a is a constant depending on θ_{001}, $\alpha_5(\tau)$ is a cosine series involving only even multiples of τ, and $\gamma_5(\tau)$ involves only sines of odd multiples of τ.

From the initial conditions we must have at $\tau = 0$

$$\rho_{001} = 0, \qquad \sigma_{001} = 0, \qquad \rho'_{001} = 0, \qquad \sigma'_{001} = 0.$$

Determining the constants of integration so as to satisfy these conditions we have the solution

$$(146)\quad \begin{aligned}\rho_{001} &= A^{(1)}_{001}\alpha_2(\tau) + A^{(2)}_{001}[\alpha_4(\tau) + \tau\alpha_3(\tau)] + \alpha_6(\tau),\\ \sigma_{001} &= A^{(1)}_{001}\gamma_1(\tau) + A^{(2)}_{001}[\gamma_3(\tau) + \tau\gamma_4(\tau)] + \gamma_6(\tau),\end{aligned}$$

where

$$A^{(1)}_{001} = \frac{1}{\Delta}[\alpha_4\beta'_6 - \alpha_6(\beta_3 + \beta'_4)],$$

$$A^{(2)}_{001} = \frac{1}{\Delta}[\alpha_6\beta'_1 - \alpha_1\beta'_6],$$

$$\alpha_6(\tau) = \alpha_5(\tau) - a\alpha_4(\tau), \qquad \gamma_6(\tau) = \gamma_5(\tau) - a\gamma_3(\tau).$$

It will be seen later that the value of $\alpha_6(\tau)$ for $\tau = 0$ plays an important rôle, so that it is necessary for us to verify that it does not vanish. By hypothesis $\alpha_5(\tau)$ is the periodic part of the particular solution for ρ in the differential equations (144). Let us put in these equations

$$\rho_{001} = \phi + a\tau\alpha_3(\tau), \qquad \sigma_{001} = \psi + a\tau\gamma_4(\tau),$$

where ϕ and ψ are the periodic parts of the particular solution. We find

$$\phi'' + \theta_2\phi + \theta_3\psi = -2a\alpha'_3(\tau) + 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2\cos 2\tau]\mu^2 + \cdots,$$

$$\psi'' + \theta_4\psi + \theta_3\phi = -2a\beta'_3(\tau);$$

or,

$$\phi'' + \theta_2\phi + \theta_3\psi = 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - 2(a_0 + \tfrac{3}{4})\beta^2\cos 2\tau]\mu^2,$$

$$\psi'' + \theta_4\psi + \theta_3\psi = -2a_0\beta\sin\tau\mu + \cdots,$$

where we have substituted

$$a = a_0 + a_2\mu^2 + \cdots.$$

The functions ϕ and ψ can be expanded as series of the form

$$\phi = \phi_0 + \phi_2\mu^2 + \cdots, \qquad \psi = \psi_1\mu + \psi_3\mu^3 + \cdots.$$

Then we get

$$\phi_0'' + \phi_0 = 1, \qquad \text{so that} \qquad \phi_0 = 1 + c_0\cos\tau,$$

$$\psi_1'' + \psi_1 = (3 - 2a_0)\beta\sin\tau + \tfrac{3}{2}\beta c_0\sin 2\tau.$$

Since ψ_1 is periodic we must have $a_0 = \frac{3}{2}$, and then

$$\psi_1 = c_1\sin\tau - \tfrac{1}{2}\beta c_0\sin 2\tau,$$

$$\phi_2'' + \phi_2 = -\tfrac{1}{4}c_0\beta^2\cos\tau + \text{other terms}.$$

Since ϕ_2 is periodic we must have $c_0 = 0$. Therefore

$$\phi = \alpha_5(\tau) = 1 + \mu^2 P_1(\mu^2), \qquad a = \tfrac{3}{2} + \mu^2 P_2(\mu^2),$$

$$\alpha_4(\tau) = 0 + [\tfrac{3}{2}\beta^2 + \tfrac{1}{2}\beta^2\cos 2\tau]\mu^2 + \cdots.$$

Consequently

$$\alpha_6(0) = \alpha_5(0) - a\alpha_4(0) = 1 + \mu^2 P_3(\mu^2), \tag{149}$$

an expression which does not vanish identically.

Coefficient of α^2.

$$\rho_{200}'' + \theta_2\rho_{200} + \theta_3\sigma_{200} = R_{200},$$
$$\sigma_{200}'' + \theta_4\sigma_{200} + \theta_3\rho_{200} = S_{200}. \tag{150}$$

The right members of these equations have the following expressions:

$$\begin{aligned} R_{200} = {} & A_{100}^{(1)2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2] \\ S_{200} = {} & A_{100}^{(1)2}[\bar\theta_{200}\alpha_2^2 + \bar\theta_{110}\alpha_2\gamma_1 + \bar\theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\bar\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar\theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\bar\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\bar\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar\theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar\theta_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The initial conditions are

$$\rho_{200} = 0, \qquad \sigma_{200} = 0, \qquad \rho_{200}' = 0, \qquad \sigma_{200}' = 0.$$

Since equations (150) are linear, and their left members have the same form as (51), the solutions will have the form

$$
\begin{aligned}
\rho_{200} &= A\alpha_2 + B\alpha_1 + C\alpha_3 + D[\tau\alpha_3 + \alpha_4] \\
&\quad + A_{100}^{(1)^2}\phi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\phi_2(\tau) + A_{100}^{(2)^2}\phi_3(\tau), \\
\sigma_{200} &= A\gamma_1 + B\gamma_2 + C\gamma_4 + D[\tau\gamma_4 + \gamma_3] \\
&\quad + A_{100}^{(1)^2}\psi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\psi_2(\tau) + A_{100}^{(2)^2}\psi_3(\tau).
\end{aligned} \tag{151}
$$

Imposing the initial conditions we find

$$B = C = 0,$$

$$
\begin{aligned}
A = {} & \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)^2} \\
& + \frac{\psi_2'(0)\alpha_4(0) - \phi_2(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
& + \frac{\psi_3'(0)\alpha_4(0) - \phi_3(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(2)^2},
\end{aligned}
$$

$$
\begin{aligned}
D = {} & \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)^2} + \frac{\phi_2(0)\gamma_1'(0) - \psi_2'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
& + \frac{\phi_3(0)\gamma_1'(0) - \psi_3'(0)\alpha_2(0)}{\Delta} A_{100}^{(2)^2}.
\end{aligned}
$$

Substituting these values in (151) we have

$$
\begin{aligned}
\rho_{200} &= A_{100}^{(1)^2}x_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}x_2(\tau) + A_{100}^{(2)^2}x_3(\tau), \\
\sigma_{200} &= A_{100}^{(1)^2}y_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}y_2(\tau) + A_{100}^{(2)^2}y_3(\tau),
\end{aligned} \tag{152}
$$

where

$$
\begin{aligned}
x_1(\tau) = {} & \phi_1(\tau) + \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4'(0) + \gamma_3'(0)]}{\Delta}\alpha_2(\tau) \\
& + \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\alpha_4(\tau) + \tau\alpha_3(\tau)],
\end{aligned}
$$

$$
\begin{aligned}
y_1(\tau) = {} & \psi_1(\tau) + \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta}\gamma_1(\tau) \\
& + \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\gamma_3(\tau) + \tau\gamma_4(\tau)],
\end{aligned}
$$

and similar expressions for x_2, y_2 and x_3, y_3, which we shall find later [equation (172)] do not interest us. The characters of x_1 and y_1 are known with the exception of ϕ_1 and ψ_1 which we will now investigate. The functions ϕ_1 and ψ_1

are those portions of the solution of the differential equations (150) which depend upon the coefficients of $A_{100}^{(1)2}$. These coefficients are homogeneous of the second degree in $\alpha_2(\tau)$ and $\gamma_1(\tau)$. In R_{200} and S_{200} the expressions

θ_{200}, $\bar{\theta}_{110}$ and θ_{020} contain only cosines of even multiples of τ;

$\bar{\theta}_{200}$, θ_{110} and $\bar{\theta}_{020}$ contain only sines of odd multiples of τ;

$\alpha_2(\tau)$ has the form $\alpha_2 = \sum_k a_k \cos[(2k+1) \pm \lambda]\tau$;

$\gamma_1(\tau)$ has the form $\gamma_1 = \sum_k b_k \sin[2k \pm \lambda]\tau$.

Consequently, so far as the coefficients of $A_{100}^{(1)2}$ are concerned, R_{200} and S_{200} have the form

$$R_{200} = \{\sum_k a_k^{(1)} \cos 2kt + \sum_k a_k^{(2)} \cos[2k \pm 2\lambda]\tau\} A_{100}^{(1)2} + \cdots,$$

$$S_{200} = \{\sum_k b_k^{(1)} \sin(2k+1)\tau + \sum_k b_k^{(2)} \sin[(2k+1) \pm 2\lambda]\tau\} A_{100}^{(1)2} + \cdots.$$

By Theorem II terms involving 2λ give rise only to periodic terms in the solution whose period is $2K\pi$. By Theorem V those parts of ρ_{200} and σ_{200} depending upon the terms in R_{200} and S_{200} which are independent of λ have the form

$$\rho = p_1(\tau) + c\tau\alpha_3(\tau), \qquad \sigma = p_2(\tau) + c\tau\gamma_4(\tau)$$

respectively, where p_1 and p_2 are periodic with the period 2π. Consequently the characters of $x_1(\tau)$ and $y_1(\tau)$ are

$$(153) \qquad x_1(\tau) = \varphi_1(\tau) + c_1\tau\alpha_3(\tau), \qquad y_1(\tau) = \varphi_2(\tau) + c_1\tau\gamma_4(\tau),$$

where $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are periodic with the period $2K\pi$.

Coefficient of $\alpha\delta$.

$$(154) \qquad \rho''_{110} + \theta_2\rho_{110} + \theta_3\sigma_{110} = R_{110}, \qquad \sigma''_{110} + \theta_4\sigma_{110} + \theta_3\rho_{110} = S_{110},$$

where

$$\begin{aligned} R_{110} &= 2A_{100}^{(1)}A_{010}^{(1)}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ &\quad + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}][2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ &\qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ &\quad + 2A_{100}^{(2)}A_{010}^{(2)}[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2], \\ S_{110} &= 2A_{100}^{(1)}A_{010}^{(1)}[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] \\ &\quad + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}][2\bar{\theta}_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar{\theta}_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ &\qquad + 2\bar{\theta}_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ &\quad + 2A_{100}^{(2)}A_{010}^{(2)}[\bar{\theta}_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar{\theta}_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar{\theta}_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The functions R_{110} and S_{110} differ from R_{200} and S_{200} only in the constants A_{ijk}. The initial conditions impose the same conditional equations. Consequently the solutions differ only in the constants A_{ijk}, so that we can express them at once without computation:

$$(155)\quad \begin{aligned} \rho_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}x_1(\tau) + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]x_2(\tau) + 2A^{(2)}_{100}A^{(2)}_{010}x_3(\tau), \\ \sigma_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}y_1(\tau) + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]y_2(\tau) + 2A^{(2)}_{100}A^{(2)}_{010}y_3(\tau), \end{aligned}$$

where the $x_i(\tau)$ and $y_i(\tau)$ are the same functions of τ as before.

Coefficient of δ^2.

By symmetry with the coefficient of α^2 it is seen that

$$(156)\quad \begin{aligned} \rho_{020} &= A^{(1)2}_{010}x_1(\tau) + A^{(1)}_{010}A^{(2)}_{010}x_2(\tau) + A^{(2)2}_{010}x_3(\tau), \\ \sigma_{020} &= A^{(1)2}_{010}y_1(\tau) + A^{(1)}_{010}A^{(2)}_{010}y_2(\tau) + A^{(2)2}_{010}y_3(\tau). \end{aligned}$$

Coefficient of ϵ^2.

Since the coefficients of the first powers of α and δ were homogeneous in the A_{ijk} of the first degree, the coefficients of α^2, $\alpha\delta$ and δ^2 are homogeneous in the A_{ijk} of the second degree. The coefficient of the first power of ϵ is not homogeneous in the A_{ijk}, hence the second power is not homogeneous. But if the functions $\alpha_6(\tau)$ and $\gamma_6(\tau)$ were zero the coefficient of the first power of ϵ would be homogeneous. By symmetry therefore, we can at once write down the terms involving the A_{ijk} to the second degree. To these must be added terms in the first degree and one term independent of the A_{ijk}.

The differential equations are

$$(157)\quad \begin{aligned} &\rho''_{002} + \theta_2\rho_{002} + \theta_3\sigma_{002} = R_{002}, \\ &\sigma''_{002} + \theta_4\sigma_{002} + \theta_3\rho_{002} = S_{002}; \\ &R_{002} = \theta_{200}\rho^2_{001} + \theta_{110}\rho_{001}\sigma_{001} + \theta_{020}\sigma^2_{001} + \theta_{101}\rho_{001}, \\ &S_{002} = \bar\theta_{200}\rho^2_{001} + \bar\theta_{110}\rho_{001}\sigma_{001} + \bar\theta_{020}\sigma^2_{001}. \end{aligned}$$

The terms involved in these expressions are shown in the following table:

R_{002}

Term	$A^{(1)2}_{001}$	$A^{(1)}_{001}A^{(2)}_{001}$	$A^{(2)2}_{001}$	$A^{(1)}_{001}$	$A^{(2)}_{001}$	1	Multiplied by
ρ^2_{001}	a_2^2	$2a_2(\tau a_3 + a_4)$	$(\tau a_3 + a_4)^2$	$2a_2a_6$	$2a_6(\tau a_3 + a_4)$	a_6^2	θ_{200}
$\rho_{001}\sigma_{001}$	$a_2\gamma_1$	$\gamma_1(\tau a_3 + a_4) + a_2(\tau\gamma_4 + \gamma_3)$	$(\tau a_3 + a_4)(\tau\gamma_4 + \gamma_3)$	$\gamma_1a_6 + a_2\gamma_6$	$a_6(\tau\gamma_4 + \gamma_3) + \gamma_6(\tau a_3 + a_4)$	$a_6\gamma_6$	θ_{110}
σ^2_{001}	γ_1^2	$2\gamma_1(\tau\gamma_4 + \gamma_3)$	$(\tau\gamma_4 + \gamma_3)^2$	$2\gamma_1\gamma_6$	$2(\tau\gamma_4 + \gamma_3)$	γ_6^2	θ_{020}
ρ_{001}				a_2	$(\tau a_3 + a_4)$	a_6	θ_{101}

In order to obtain the S_{002} it is necessary in the above table only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$.

The solutions of equations (157) may be expressed in the form

$$\begin{aligned}\rho_{002} &= A_{001}^{(1)^2} x_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} x_2(\tau) + A_{001}^{(2)^2} x_3(\tau) + A_{001}^{(1)} x_4(\tau) + A_{001}^{(2)} x_5(\tau) + x_6(\tau),\\ \sigma_{002} &= A_{001}^{(1)^2} y_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} y_2(\tau) + A_{001}^{(2)^2} y_3(\tau) + A_{001}^{(1)} y_4(\tau) + A_{001}^{(2)} y_5(\tau) + y_6(\tau).\end{aligned} \tag{158}$$

The coefficients of $A_{001}^{(1)}$ in the differential equations are homogeneous of the first degree in α_2 and γ_1, every term of which involves the first multiple of $\lambda\tau$. Hence the solutions for these terms by Theorem II involve non-periodic terms, and we can write

$$\begin{aligned}x_4(\tau) &= \varphi_3(\tau) + c_2\tau\alpha_1(\tau) + c_3\tau\alpha_3(\tau),\\ y_4(\tau) &= \varphi_4(\tau) + c_2\tau\gamma_2(\tau) + c_3\tau\gamma_4(\tau),\end{aligned} \tag{159}$$

where $\varphi_3(\tau)$ and $\varphi_4(\tau)$ are periodic with the period $2K\pi$. As is seen from the table, $x_6(\tau)$ and $y_6(\tau)$ do not involve the λ. They have therefore the form

$$\begin{aligned}x_6(\tau) &= \varphi_5(\tau) + c_4\tau\alpha_3(\tau),\\ y_6(\tau) &= \varphi_6(\tau) + c_4\tau\gamma_4(\tau).\end{aligned} \tag{160}$$

It will be verified in (172) that we do not need to know the character of x_5 and y_5.

Coefficient of $\alpha\epsilon$.

The differential equations are

$$\begin{aligned}\rho_{101}'' + \theta_2\rho_{101} + \theta_3\sigma_{101} &= R_{101},\\ \sigma_{101}'' + \theta_4\sigma_{101} + \theta_3\rho_{101} &= S_{101}.\end{aligned} \tag{161}$$

$$R_{101} = \theta_{200}[2\rho_{100}\rho_{001}] + \theta_{110}[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}] + \theta_{020}[2\sigma_{100}\sigma_{001}] + \theta_{101}\rho_{100},$$

$$S_{101} = \bar{\theta}_{200}[2\rho_{100}\rho_{001}] + \bar{\theta}_{110}[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}] + \bar{\theta}_{020}[2\sigma_{100}\sigma_{001}].$$

The following table shows the character of the terms entering into these expressions:

R_{101}

	$2A_{100}^{(1)}A_{001}^{(1)}$	$A_{100}^{(1)}A_{001}^{(2)} + A_{100}^{(2)}A_{001}^{(1)}$	$2A_{100}^{(2)}A_{001}^{(2)}$	$A_{100}^{(1)}$	$A_{100}^{(2)}$	Multiplied by
$\rho_{100}\rho_{001}$	α_2^2	$2\alpha_2(\tau\alpha_3+\alpha_4)$	$(\tau\alpha_3+\alpha_4)^2$	$2\alpha_2\gamma_6$	$2\alpha_6(\tau\alpha_3+\alpha_4)$	θ_{200}
$\rho_{100}\sigma_{001}$ $+\rho_{001}\sigma_{100}$	$\alpha_2\gamma_1$	$\gamma_1(\tau\alpha_3+\alpha_4)$ $+\alpha_2(\tau\gamma_4+\gamma_3)$	$(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)$	$\alpha_6\gamma_1+\alpha_2\gamma_6$	$\alpha_6(\tau\gamma_4+\gamma_3)$ $+\gamma_6(\tau\alpha_3+\alpha_4)$	θ_{110}
$\sigma_{100}\sigma_{001}$	γ_1^2	$2\gamma_1(\tau\gamma_4+\gamma_3)$	$(\tau\gamma_4+\gamma_3)^2$	$2\gamma_1\gamma_6$	$2\gamma_6(\tau\gamma_4+\gamma_3)$	θ_{020}
ρ_{100}				α_2	$(\tau\alpha_3+\alpha_4)$	θ_{101}

In order to obtain S_{101} it is necessary only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$. This table shows that R_{101} and S_{101} differ from R_{002} and S_{002} only in

the constants A_{ijk}. Since the initial conditions impose the same conditional equations as for the coefficient of ϵ^2 we can at once write down the solution

$$\begin{aligned}\rho_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}x_1(\tau) + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]x_2(\tau)\\ &\quad + 2A^{(2)}_{100}A^{(2)}_{001}x_3(\tau) + A^{(1)}_{100}x_4(\tau) + A^{(2)}_{100}x_5(\tau),\\ \sigma_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}y_1(\tau) + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]y_2(\tau)\\ &\quad + 2A^{(2)}_{100}A^{(2)}_{001}y_3(\tau) + A^{(1)}_{100}y_4(\tau) + A^{(2)}_{100}y_5(\tau).\end{aligned} \tag{162}$$

Coefficient of $\delta\epsilon$.

This coefficient can be obtained by symmetry from the coefficient of $\alpha\epsilon$ by permutation of the first and second subscripts of the A_{ijk}. Therefore

$$\begin{aligned}\rho_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}x_1(\tau) + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]x_2(\tau)\\ &\quad + 2A^{(2)}_{010}A^{(2)}_{001}x_3(\tau) + A^{(1)}_{010}x_4(\tau) + A^{(2)}_{010}x_5(\tau),\\ \sigma_{101} &= 2A^{(1)}_{010}A^{(1)}_{001}y_1(\tau) + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]y_2(\tau)\\ &\quad + 2A^{(2)}_{010}A^{(2)}_{001}y_3(\tau) + A^{(1)}_{010}y_4(\tau) + A^{(2)}_{010}y_5(\tau).\end{aligned} \tag{163}$$

This concludes the computation of all terms up to the second order inclusive in α, δ and ϵ. In order to establish the existence of the periodic solutions it is not necessary to carry the computation further.

§ 16. *Existence of periodic orbits reëntrant only after many revolutions.*

We have chosen the initial conditions so that at $\tau = 0$ the particle is crossing the ρ-axis orthogonally. It is obvious geometrically that if at any future time it again crosses the ρ-axis perpendicularly the orbit will be a closed one and the motion in it will be periodic. The conditions that the particle shall cross the ρ-axis perpendicularly at $\tau = T$ is that at this epoch

$$\rho' = \sigma = 0.$$

The equations of variation have the period $2K\pi$. Therefore we shall choose $T = K\pi$. Since ρ is an even series in τ, and σ is an odd series, all the purely periodic terms in ρ' and σ are sines and consequently vanish at $\tau = K\pi$. The terms which do not vanish must carry τ as a factor. The conditions for periodicity give us two equations, namely, at $\tau = K\pi$,

$$\begin{aligned}\rho' = 0 &= a_{100}\alpha + a_{010}\delta + a_{001}\epsilon + a_{200}\alpha^2 + a_{110}\alpha\delta\\ &\quad + a_{020}\delta^2 + a_{101}\alpha\epsilon + a_{011}\delta\epsilon + a_{002}\epsilon^2 + \cdots,\\ \sigma = 0 &= b_{100}\alpha + b_{010}\delta + b_{001}\epsilon + b_{200}\alpha^2 + b_{110}\alpha\delta\\ &\quad + b_{020}\delta^2 + b_{101}\alpha\epsilon + b_{011}\delta\epsilon + b_{002}\epsilon^2 + \cdots,\end{aligned} \tag{164}$$

where a_{ijk} and b_{ijk} are the values derived from the series just computed. Their values are as follows:

$$
\begin{aligned}
&a_{100} = A^{(2)}_{100} u, \qquad a_{010} = A^{(2)}_{010} u, \qquad a_{001} = A^{(2)}_{001} u,\\
&b_{100} = A^{(2)}_{100} v, \qquad b_{010} = A^{(2)}_{010} v, \qquad b_{001} = A^{(2)}_{001} v,
\end{aligned} \tag{165}
$$

$$
\begin{aligned}
a_{200} &= A^{(1)2}_{100}\bar{x}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{x}_2 + A^{(2)2}_{100}\bar{x}_3,\\
b_{200} &= A^{(1)2}_{100}\bar{y}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{y}_2 + A^{(2)2}_{100}\bar{y}_3,\\
a_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{x}_3,\\
b_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{y}_3,\\
a_{020} &= A^{(1)2}_{010}\bar{x}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{x}_2 + A^{(2)2}_{010}\bar{x}_3,\\
b_{020} &= A^{(1)2}_{010}\bar{y}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{y}_2 + A^{(2)2}_{010}\bar{y}_3,\\
a_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{100}\bar{x}_4 + A^{(2)}_{100}\bar{x}_5,\\
b_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{100}\bar{y}_4 + A^{(2)}_{100}\bar{y}_5,\\
a_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{010}\bar{x}_4 + A^{(2)}_{010}\bar{x}_5,\\
b_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{010}\bar{y}_4 + A^{(2)}_{010}\bar{y}_5,\\
a_{002} &= A^{(1)2}_{001}\bar{x}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{x}_2 + A^{(2)2}_{001}\bar{x}_3 + A^{(1)}_{001}\bar{x}_4 + A^{(2)}_{001}\bar{x}_5 + \bar{x}_6,\\
b_{002} &= A^{(1)2}_{001}\bar{y}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{y}_2 + A^{(2)2}_{001}\bar{y}_3 + A^{(1)}_{001}\bar{y}_4 + A^{(2)}_{001}\bar{y}_5 + \bar{y}_6,
\end{aligned}
$$

where

$$
\begin{aligned}
&u = K\pi \frac{d\alpha_3}{d\tau}, \quad v = k\pi\gamma_1,\\
&\bar{x}_i = \frac{dx_i}{d\tau}, \qquad \bar{y}_i = y_i, \qquad \text{at } \tau = K\pi.
\end{aligned} \tag{166}
$$

Let us solve now the equation (164), $\rho' = 0$, for ϵ as a power series in α and δ. We obtain

$$
\epsilon = \epsilon_{10}\alpha + \epsilon_{01}\delta + \epsilon_{20}\alpha^2 + \epsilon_{11}\alpha\delta + \epsilon_{02}\delta^2 + \cdots, \tag{167}
$$

where the coefficients ϵ_{ij} have the following values:

$$
\epsilon_{10} = -\frac{a_{100}}{a_{001}},
$$

$$
\epsilon_{01} = -\frac{a_{010}}{a_{001}},
$$

$$
\epsilon_{20} = -\frac{a_{200}a_{001}^2 - a_{100}a_{101}a_{001} + a_{002}a_{100}^2}{a_{001}^3},
$$

$$\epsilon_{11} = -\frac{a_{110}a_{001}^2 - a_{011}a_{100}a_{001} - a_{101}a_{010}a_{001} + 2a_{002}a_{100}a_{010}}{a_{001}^3},$$

$$\epsilon_{02} = -\frac{a_{020}a_{001}^2 - a_{011}a_{010}a_{001} + a_{002}a_{010}^2}{a_{001}^3}.$$

Solving now equation (164), $\sigma = 0$, for ϵ in terms of α and δ, we obtain

$$\epsilon = \bar{\epsilon}_{10}\alpha + \bar{\epsilon}_{01}\delta + \bar{\epsilon}_{20}\alpha^2 + \bar{\epsilon}_{11}\alpha\delta + \bar{\epsilon}_{02}\delta^2 + \cdots, \tag{168}$$

where the $\bar{\epsilon}_{ij}$ have the same expressions in the b_{klm} as the ϵ_{ij} have in the a_{klm}. Subtracting (167) from (168), we have

$$0 = [\bar{\epsilon}_{10} - \epsilon_{10}]\alpha + [\bar{\epsilon}_{01} - \epsilon_{01}]\delta + [\bar{\epsilon}_{20} - \epsilon_{20}]\alpha^2 + [\bar{\epsilon}_{11} - \epsilon_{11}]\alpha\delta + [\bar{\epsilon}_{02} - \epsilon_{02}]\delta^2 + \cdots. \tag{169}$$

We must now examine the coefficients of this series.

$$\begin{aligned}[\bar{\epsilon}_{10} - \epsilon_{10}] &= \frac{a_{100}}{a_{001}} - \frac{b_{100}}{b_{001}} = \frac{A_{100}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{100}^{(2)}v}{A_{001}^{(2)}v} = 0,\\ [\bar{\epsilon}_{01} - \epsilon_{10}] &= \frac{a_{010}}{a_{001}} - \frac{b_{010}}{b_{001}} = \frac{A_{010}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{010}^{(2)}v}{A_{001}^{(2)}v} = 0.\end{aligned} \tag{170}$$

Both of the linear terms therefore vanish. The computation of the second degree terms is somewhat more complicated. It simplifies matters somewhat if we observe that the $\bar{\epsilon}_{ij}$ have the same expressions in v and the $\bar{y}_i$ as the ϵ_{ij} have in u and $\bar{x}_i$. It is sufficient therefore to compute one and derive the other from it. From (165) we have

$$-\epsilon_{20} = \frac{a_{200}}{a_{001}} - \frac{a_{100}a_{101}}{a_{001}^2} + \frac{a_{002}a_{100}^2}{a_{001}^3}.$$

Substituting in these fractions the values of the a_{ijk} from (165) we get

$$\begin{aligned}\frac{a_{200}}{a_{001}} &= \frac{1}{A_{001}^{(2)3}}\left[A_{100}^{(1)2}A_{001}^{(2)2}\frac{\bar{x}_1}{u} + A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)2}\frac{\bar{x}_2}{u} + A_{100}^{(2)2}A_{001}^{(2)2}\frac{\bar{x}_3}{u}\right],\\ -\frac{a_{100}a_{101}}{a_{001}^2} &= \frac{1}{A_{001}^{(2)3}}\left[-2A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_1}{u} + [-A_{100}^{(2)2}A_{001}^{(1)}A_{001}^{(2)} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)2}]\frac{\bar{x}_2}{u}\right.\\ &\qquad \left. - 2A_{100}^{(2)2}A_{001}^{(2)2}\frac{\bar{x}_3}{u} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)}\frac{\bar{x}_4}{u} - A_{100}^{(2)2}A_{001}^{(2)}\frac{\bar{x}_5}{u}\right],\\ \frac{a_{002}a_{100}^2}{a_{001}^3} &= \frac{1}{A_{001}^{(2)3}}\left[A_{100}^{(2)2}A_{001}^{(1)2}\frac{\bar{x}_1}{u} + A_{100}^{(2)2}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_2}{u} + A_{100}^{(2)2}A_{001}^{(2)2}\frac{\bar{x}_3}{u}\right.\\ &\qquad \left. + A_{100}^{(2)2}A_{001}^{(1)}\frac{\bar{x}_4}{u} + A_{100}^{(2)2}A_{001}^{(2)}\frac{\bar{x}_5}{u}A_{100}^{(2)2}\frac{\bar{x}_6}{u}\right].\end{aligned} \tag{171}$$

For the sum of these three expressions, there results

$$-\epsilon_{20}=\frac{1}{A_{001}^{(2)3}}\Big[[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]^2\frac{\bar{x}_1}{u} \\ -[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]A_{100}^{(2)}\frac{\bar{x}_4}{u}+A_{100}^{(2)2}\frac{\bar{x}_6}{u}\Big], \tag{172}$$

the coefficients of $\bar{x}_2/u$, $\bar{x}_3/u$ and $\bar{x}_5/u$ vanishing identically. This is the reason why it was not necessary to compute x_2, x_3, and x_5 in (152) and (158).

Changing the $\bar{x}_i$ into $\bar{y}_i$ and u into v gives us $-\bar{\epsilon}_{20}$. Hence

$$[\bar{\epsilon}_{20}-\epsilon_{20}]=\frac{1}{A_{001}^{(2)3}}\Big\{[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]^2\Big[\frac{\bar{x}_1}{u}-\frac{\bar{y}_1}{v}\Big] \\ -A_{100}^{(2)}[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]\Big[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\Big]+A_{100}^{(2)2}\Big[\frac{\bar{x}_6}{u}-\frac{\bar{y}_6}{v}\Big]\Big\}. \tag{173}$$

But $[\bar{x}_1/u-\bar{y}_1/v]$ and $[\bar{x}_6/u-\bar{y}_6/v]$ both vanish since

$$\bar{x}_1=c_1u, \qquad \bar{y}_1=c_1v, \qquad \bar{x}_6=c_4u, \qquad \bar{y}_6=c_4v,$$

as is readily seen from (153) and (160). It is also seen on referring to (159) that $[\bar{x}_4/u-\bar{y}_4/v]$ does not vanish, but is equal to

$$\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}=c_2\Big[\frac{1}{u}\frac{d\alpha_1}{d\tau}-\frac{\gamma_2}{v}\Big]_{\tau=K\pi}.$$

Hence

$$[\bar{\epsilon}_{20}-\epsilon_{20}]=-\frac{A_{100}^{(2)}[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]}{A_{001}^{(2)3}}\Big[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\Big]. \tag{174}$$

Without repeating the details of the computation we find in a similar way

$$[\bar{\epsilon}_{11}-\epsilon_{11}]=-\frac{A_{010}^{(2)}[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}]+A_{100}^{(2)}[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}]}{A_{001}^{(2)3}}\Big[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\Big], \tag{175}$$

$$[\bar{\epsilon}_{02}-\epsilon_{02}]=-\frac{A_{010}^{(2)}[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}]}{A_{001}^{(2)3}}\Big[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\Big]. \tag{176}$$

Substituting these values in (169), we find that the second degree terms in α and δ are factorable. Thus (169) becomes

$$0=\frac{1}{A_{001}^{(2)3}}\Big[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\Big][A_{100}^{(2)}\alpha+A_{010}^{(2)}\delta+\cdots][(A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)})\alpha \\ +(A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)})\delta+\cdots]. \tag{177}$$

There are therefore two real solutions for δ as power series in α. Substituting these values of δ in (168) we find the two corresponding values of ϵ. In this manner we find:

First solution.

$$\delta = -\frac{A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}{A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}\alpha + \cdots = \frac{\gamma'_5 - a\gamma'_3}{\alpha_5 - a\alpha_4}\alpha + \cdots = \frac{\gamma'_6(0)}{\alpha_6(0)}\alpha + \cdots,$$

(178)

$$\epsilon = \frac{A^{(1)}_{100}A^{(2)}_{010} - A^{(2)}_{100}A^{(1)}_{010}}{A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}\alpha + \cdots = \frac{1}{\alpha_5 - a\alpha_4}\alpha + \cdots = \frac{1}{\alpha_6(0)}\alpha + \cdots.$$

Second solution.

$$\delta = -\frac{A^{(2)}_{100}}{A^{(2)}_{010}}\alpha + \cdots = \frac{\gamma'_1(0)}{\alpha_2(0)}\alpha + \cdots = \gamma'_2(0)\cdot\alpha + \cdots,$$

(179)

$$\epsilon = -\frac{A^{(2)}_{100}}{A^{(2)}_{001}}\alpha - \frac{A^{(2)}_{010}}{A^{(2)}_{001}}\delta + \cdots = 0\cdot\alpha + \cdots,$$

where α_6 and γ_6 are the quantities defined in (147), and $\gamma'_6(0)$ is the value of $d\gamma_6/d\tau$ for $\tau = 0$. Thus one solution for ϵ begins with the first power of α while the other certainly does not begin before the second, but in both solutions δ begins with the first power of α.

§ 17. *Construction of the solutions having the period* $2K\pi$.

We have just proved the existence of series for ρ, σ and ϵ proceeding in powers of the initial value of ρ (let us call this value e).* The series for ρ and σ are periodic in τ with the period $2K\pi$, and since this condition holds for all values of e sufficiently small each coefficient is separately periodic. The series for ρ is even in τ, and the series for σ is odd in τ. These series have the form

$$\rho = \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \cdots,$$

(180)

$$\sigma = \sigma_1 e + \sigma_2 e^2 + \sigma_3 e^3 + \cdots,$$

$$\epsilon = \epsilon_1 e + \epsilon_2 e^2 + \epsilon_3 e^3 + \cdots.$$

We may substitute these series in the differential equations and integrate the coefficients of each power of e step by step, and determine the constants in such a way that ρ and σ shall be periodic, and satisfy the initial conditions

(181) $$\rho(0) = e, \qquad \sigma(0) = 0, \qquad \rho'(0) = 0, \qquad \sigma'(0) = \delta,$$

where δ is a constant at present unknown but to be determined in the process.

Substituting the series (180) in the differential equations (138), we find for the coefficient of the first power of e

(182) $$\rho_1'' + \theta_2\rho_1 + \theta_3\sigma_1 = \theta_{001}\epsilon_1, \qquad \sigma_1'' + \theta_4\sigma_1 + \theta_3\rho_1 = 0.$$

* The reason for changing α to e is that this parameter corresponds to the eccentricity in the two-body problem.

Since by the condition of orthogonality ρ must be even in τ and σ odd in τ, the solution complying with the condition is

$$\begin{aligned}\rho_1 &= A^{(1)}\alpha_2(\tau) + D^{(1)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_1[a\tau\alpha_3(\tau) + \alpha_5(\tau)],\\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + D^{(1)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_1[a\tau\gamma_4(\tau) + \gamma_5(\tau)],\end{aligned} \tag{183}$$

where $\alpha_5(\tau)$ contains only cosines of even multiples of τ and $\gamma_5(\tau)$ contains only sines of odd multiples of τ.

In order that this solution shall be periodic it is necessary and sufficient that

$$D^{(1)} = -a\epsilon_1.$$

Upon imposing this condition the solution (183) becomes [see eq. (146)]

$$\begin{aligned}\rho_1 &= A^{(1)}\alpha_2(\tau) + \epsilon_1[\alpha_5(\tau) - a\alpha_4(\tau)] = A^{(1)}\alpha_2(\tau) + \epsilon_1\alpha_6(\tau),\\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + \epsilon_1[\gamma_5(\tau) - a\gamma_3(\tau)] = A^{(1)}\gamma_1(\tau) + \epsilon_1\gamma_6(\tau).\end{aligned} \tag{184}$$

It remains to impose the initial condition that $\rho_1 = 1$ at $\tau = 0$. From this condition we get

$$1 = A^{(1)}\bar{\alpha}_2 + \epsilon_1\bar{\alpha}_6, \tag{185}$$

where $\bar{\alpha}_2$ and $\bar{\alpha}_6$ denote the values of these functions when $\tau = 0$.

Coefficient of e^2.

$$\begin{aligned}\rho_2'' + \theta_2\rho_2 + \theta_3\sigma_2 &= \theta_{001}\epsilon_2 + \theta_{101}\epsilon_1\rho_1 + \theta_{200}\rho_1^2 + \theta_{110}\rho_1\sigma_1 + \theta_{020}\sigma_1^2 = R_2,\\ \sigma_2'' + \theta_4\sigma_2 + \theta_3\rho_2 &= \qquad\qquad \bar{\theta}_{200}\rho_1^2 + \bar{\theta}_{110}\rho_1\sigma_1 + \bar{\theta}_{020}\sigma_1^2 = S_2.\end{aligned} \tag{186}$$

Every term of R_2 and S_2 contains either $A^{(1)}$, ϵ_1 or ϵ_2 as a factor. Arranged in this manner we have the expansions

$$\begin{aligned}R_2 = {} & A^{(1)2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2]\\ & + A^{(1)}\epsilon_2[2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\theta_{020}\gamma_1\gamma_6 + \theta_{101}\alpha_2]\\ & + \epsilon_1^2[\theta_{200}\alpha_6^2 + \theta_{110}\alpha_6\gamma_6 + \theta_{020}\gamma_6^2 + \theta_{101}\alpha_6] + \epsilon_2[\theta_{001}],\\ S_2 = {} & A^{(1)2}[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] + A^{(1)}\epsilon_1[2\bar{\theta}_{200}\alpha_2\alpha_6 + \bar{\theta}_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\bar{\theta}_{020}\gamma_1\gamma_6]\\ & + \epsilon_1^2[\bar{\theta}_{200}\alpha_6^2 + \bar{\theta}_{110}\alpha_6\gamma_6 + \bar{\theta}_{020}\gamma_6^2].\end{aligned}$$

In order to understand the character of the solution of these equations we must examine the character of the various terms. The coefficient of $A^{(1)2}$ in both R_2 and S_2 is homogeneous of the second degree in α_2 and γ_1. Its expansion therefore involves terms carrying $2\lambda\tau$ and terms independent of λ. By Theorem II the solution for the terms in $2\lambda\tau$ is periodic. The terms independent of λ are cosines of even multiples of τ in R_2 and sines of odd multiples of τ in

S_2. These terms have the same character as those in the coefficients of ϵ_1^2 and ϵ_2 and may be considered under the discussion of those terms.

The coefficients of $A^{(1)}\epsilon_1$ in both R_2 and S_2 are homogeneous of the first degree in α_2 and γ_1, all terms of which carry the first multiple of $\lambda\tau$. By Theorem VII the solution for ρ_2 will carry the term $\tau\alpha_1(\tau)$, and for σ_2 the term $\tau\gamma_2(\tau)$. Non-periodic terms of this character do not arise elsewhere in the solution, hence, in order to avoid them, we must take either $A^{(1)}=0$ or $\epsilon_1=0$. If we choose $A^{(1)}=0$, then, by (185), ϵ_1 is determined so that we must have $\epsilon_1=1/\bar{\alpha}_6$, thus agreeing with the first solution (178) of the existence proof. But if we choose $\epsilon_1=0$ so that by (185) $A^{(1)}=1/\bar{\alpha}_2$ we are in agreement with the second solution (179) of the existence proof. We will commence by developing the first solution. This necessitates the choice $A^{(1)}=0$, $\epsilon_1=1/\bar{\alpha}_6$.

First solution.

Since $A^{(1)}=0$ all terms in R_2 and S_2 which carry $\lambda\tau$ or any multiple of it vanish. There remains

$$(187)\qquad \begin{aligned} R_2&=\epsilon_1^2\left[\theta_{200}\alpha_6^2+\theta_{110}\alpha_6\gamma_6+\theta_{020}\gamma_6^2+\theta_{101}\alpha_6\right]+\epsilon_2\theta_{001},\\ S_2&=\epsilon_1^2\left[\bar{\theta}_{200}\alpha_6^2+\bar{\theta}_{110}\alpha_6\gamma_6+\bar{\theta}_{020}\gamma_6^2\right].\end{aligned}$$

We have also

$$(188)\qquad \rho_1=\frac{\alpha_6(\tau)}{\bar{\alpha}_6},\qquad \sigma_1=\frac{\gamma_6(\tau)}{\bar{\alpha}_6},\qquad \epsilon_1=\frac{1}{\bar{\alpha}_6}.$$

It is easy to characterize R_2 and S_2. R_2 contains only cosines of even multiples of τ, and S_2 contains only sines of odd multiples of τ. Since ρ_2 is even in τ and σ_2 odd, the solution satisfying this condition is

$$\rho_2=A^{(2)}\alpha_2(\tau)+D^{(2)}[\tau\alpha_3(\tau)+\alpha_4(\tau)]+[\eta_2(\tau)+a_2\tau\alpha_3(\tau)]+\epsilon_2[\alpha_5(\tau)+a\tau\alpha_3(\tau)],$$

$$\sigma_2=A^{(2)}\gamma_1(\tau)+D^{(2)}[\tau\gamma_4(\tau)+\gamma_3(\tau)]+[\zeta_2(\tau)+a_2\tau\gamma_4(\tau)]+\epsilon_2[\gamma_5(\tau)+a\tau\gamma_4(\tau)].$$

In this solution the terms are grouped according to their origin. The first two terms are the complementary function. The third arises from the terms carrying ϵ_1^2 as a factor. The fourth arises from the terms carrying ϵ_2 as a factor. a_2 is a constant depending upon the coefficients of ϵ_1^2 in the differential equations. $\alpha_5(\tau)$ and $\gamma_5(\tau)$ are the same functions as in the coefficient of the first power of e. By Theorems IV and V, $\eta_2(\tau)$ and $\zeta_2(\tau)$ are periodic functions of τ with the period 2π and so constituted that $\eta_2(\tau)$ contains only cosines of even multiples of τ, and $\zeta_2(\tau)$ contains only sines of odd multiples.

In order that ρ_2 and σ_2 shall be periodic we must have

$$D^{(2)}=-a_2-a\epsilon_2,$$

and this makes

$$\rho_2 = A^{(2)}\alpha_2(\tau) + \epsilon_2\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau),$$

$$\sigma_2 = A^{(2)}\gamma_1(\tau) + \epsilon_2\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau).$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$, and this determines ϵ_2 so that

$$\epsilon_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2 - A^{(2)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

$A^{(2)}$ is not yet determined, but it is obvious that it must be zero in order to satisfy the periodicity condition on the coefficient of e^3 in which the non-periodic parts arising from terms involving the first multiple of $\lambda\tau$ carry $A^{(2)}$ as a factor. Hence the only way to avoid non-periodic terms of this character is to choose $A^{(2)} = 0$. Anticipating this step then we have

$$\rho_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau),$$

(191)

$$\sigma_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau).$$

Hence ρ_2 contains only cosines of even multiples of τ, and σ_2 contains only sines of odd multiples of τ.

It remains only to show that this process of integration can be carried on indefinitely. Assuming that up to and including ρ_{i-1} and σ_{i-1} every ρ_j and σ_j is periodic with the period 2π and that the ρ_j contain only cosines of even multiples of τ and σ_j only sines of odd multiples of τ, except that ρ_{i-1} contains the term $A^{(i-1)}\alpha_2(\tau)$ and σ_{i-1} contains the term $A^{(i-1)}\gamma_1(\tau)$, it will be shown that the same conditions obtain for the next succeeding step. For ρ_i and σ_i we have from the differential equations (138)

$$\rho_i'' + \theta_2\rho_i + \theta_3\sigma_i = \theta_{001}\epsilon_i + A^{(i-1)}[\theta_{101}\epsilon_1\alpha_2 + 2\theta_{200}\rho_1\alpha_2$$

(192)

$$+ \theta_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\theta_{030}\sigma_1\gamma_1] + \Phi_i,$$

$$\sigma_i'' + \theta_4\sigma_i + \theta_3\rho_i = A^{(i-1)}[2\bar{\theta}_{200}\rho_1\alpha_2 + \bar{\theta}_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\bar{\theta}_{020}\sigma_1\gamma_1] + \Psi_i.$$

From the properties of the differential equations it is readily seen that Φ_i contains only known terms all of which are cosines of even multiples of τ, and that Ψ_i contains only known terms all of which are sines of odd multiples of τ. The coefficients of $A^{(i-1)}$ are homogeneous of the first degree in α_2 and γ_1, and consequently each term involves a first multiple of $\lambda\tau$. Their solution gives rise to non-periodic terms of the form $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. They carry $A^{(i-1)}$ as a

factor, and since terms of this type arise nowhere else in the solution we can make them disappear only by putting $A^{(i-1)} = 0$. The solution for (192) then has the form

$$\begin{aligned}\rho_i &= A^{(i)}\alpha_2(\tau) + D^{(i)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_i(\tau) + a_i\tau\alpha_3(\tau)] + \epsilon_i[\alpha_5(\tau) + a\tau\alpha_3(\tau)],\\ \sigma_i &= A^{(i)}\gamma_1(\tau) + D^{(i)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_i(\tau) + a_i\tau\gamma_4(\tau)] + \epsilon_i[\gamma_5(\tau) + a\tau\gamma_4(\tau)],\end{aligned} \tag{193}$$

where $\eta_i(\tau)$ and $\zeta_i(\tau)$ are periodic with the period 2π, and by Theorem V, $\eta_i(\tau)$ contains only cosines of even multiples of τ, and $\zeta_i(\tau)$ contains only sines of odd multiples of τ.

For ρ_i and σ_i to be periodic it is necessary and sufficient that

$$D^{(i)} = -a_i - a\epsilon_i,$$

which makes

$$\begin{aligned}\rho_i &= A^{(i)}\alpha_2(\tau) + \eta_i(\tau) - a_i\alpha_4(\tau) + \epsilon_i\alpha_6(\tau),\\ \sigma_i &= A^{(i)}\gamma_1(\tau) + \zeta_i(\tau) - a_i\gamma_3(\tau) + \epsilon_i\gamma_6(\tau).\end{aligned} \tag{194}$$

From the initial conditions we must have $\rho_i(0) = 0$. This condition determines ϵ_i to be

$$\epsilon_i = \frac{a_i\bar{\alpha}_4 - \bar{\eta}_i - A^{(i)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

Thus the constants are uniquely determined. The ρ_i and σ_i have the properties assumed for those having lower subscripts, and the process of integration can be continued indefinitely. Every $A^{(j)}$ is zero. Since no terms involving the $\lambda\tau$ enter, the solution has the period 2π. But the orbits represented belong to the class of generating orbits from which we started. In other words, we set out with a generating orbit for which the initial distance was, let us say, r_0, and we have found another generating orbit for which the initial distance is $r_0 + e$ (e arbitrary). There is nothing surprising in this, for r_0 is a function of an arbitrary constant β. Let us suppose we had started with a definite value of β, let us say β_0. This gives us a definite generating orbit with a definite initial distance r_0. Let us seek now the generating orbit for which the initial distance is $r_0 + e$. If e is sufficiently small we can evidently give an increment ϵ to β which will increase r_0 by the amount e. We have

$$r_0 = f(\beta_0),$$

$$r_0 + e = f(\beta_0 + \epsilon).$$

Expanding the right member of the last equation by Taylor's theorem we have

$$e = \frac{\partial f}{\partial \beta_0}\epsilon + \tfrac{1}{2}\frac{\partial^2 f}{\partial \beta_0^2}\epsilon^2 + \cdots,$$

and by inversion, since

$$\left.\frac{\partial f}{\partial \beta}\right|_{\beta=\beta_0} \neq 0,$$

$$\epsilon = \frac{1}{\dfrac{\partial f}{\partial \beta_0}} e + \cdots,$$

which we might write

$$\epsilon = e_1 e + e_2 e^2 + \cdots.$$

Then by substituting in the generating orbit

$$\beta = \beta_0 + e_1 e + e_2 e^2 + \cdots$$

and arranging as a power series in e, we would obtain the generating orbit which we sought. As these are the same conditions that are imposed when we seek new orbits through the equations of variation it is to be anticipated that one of the class of generating orbits satisfies the conditions.

Second solution.

We return now to equation (186) where the two methods of satisfying the periodicity condition presented themselves, and we shall continue with the second solution. We choose $\epsilon_1 = 0$, which we found determines $A^{(1)}$ to be $A^{(1)} = 1/\bar{\alpha}_2$. From (82) it is seen that $\bar{\alpha}_2 = \alpha_2(0) = 1$, and therefore $A^{(1)} = 1$. Hence in the second solution

$$\rho_1 = \alpha_2(\tau), \qquad \sigma_1 = \gamma_1(\tau). \tag{195}$$

Using these values of $A^{(1)}$ and ϵ_1, R_2 and S_2 of (186) become

$$\begin{aligned} R_2 &= [\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] + \epsilon_2\theta_{001}, \\ S_2 &= [\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{101}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2]. \end{aligned} \tag{196}$$

All of the terms in these expressions except $\epsilon_2\theta_{001}$ are of the second degree in α_2 and γ_1. Therefore they involve terms carrying $2\lambda\tau$ and terms independent of $\lambda\tau$. θ_{001} also is independent of $\lambda\tau$. In the solution the terms depending upon $2\lambda\tau$ are periodic by Theorem II. As for the terms independent of λ, R_2 contains only cosines of even multiples of τ and S_2 contains only sines of odd multiples of τ. These terms give rise to non-periodic terms in the solution which has the form

$$\begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + D^{(2)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \phi_2(\lambda, \tau) \\ &\quad + [\eta_2(\tau) + a_2\tau\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\alpha_3(\tau)]; \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \psi_2(\lambda, \tau) \\ &\quad + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)]. \end{aligned} \tag{197}$$

In these equations $\phi_2(\lambda, \tau)$ and $\psi_2(\lambda, \tau)$ are the periodic terms involving λ, η_2 and ζ_2 are the periodic terms with the period 2π; a_2 the constant belonging to the non-periodic part, and the coefficients of ϵ_2 are the solutions arising from the coefficient of ϵ_2 in the differential equations. In order that this solution (197) shall be periodic it is necessary that

$$D^{(2)} = -a_2 - a\epsilon_2,$$

and this reduces ρ_2 and σ_2 to

$$\begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + \phi_2(\lambda, \tau) + \eta_2(\tau) - a_2\alpha_4(\tau) + \epsilon_2\alpha_6(\tau), \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + \psi_2(\lambda, \tau) + \zeta_2(\tau) - a_2\gamma_3(\tau) + \epsilon_2\gamma_6(\tau). \end{aligned} \tag{198}$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$. Hence, since $\alpha_2(0) = 1$,

$$A^{(2)} = -\bar{\phi}_2 - \bar{\eta}_2 + a_2\bar{\alpha}_4 - \epsilon_2\bar{\alpha}_6.$$

The constant ϵ_2 is determined by the periodicity condition for the coefficient of e^3.

Coefficient of e^3.

$$\rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = R_3, \qquad \sigma_3'' + \theta_4\sigma_3 + \theta_3\rho_3 = S_3, \tag{199}$$

where

$$\begin{aligned} R_3 = {} & \theta_{001}\epsilon_3 + \theta_{101}\epsilon_2\rho_1 + 2\theta_{200}\rho_1\rho_2 + \theta_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\theta_{020}\sigma_1\sigma_2 + \theta_{300}\rho_1^3 \\ & + \theta_{210}\rho_1^2\sigma_1 + \theta_{120}\rho_1\sigma_1^2 + \theta_{030}\sigma_1^3, \\ S_3 = {} & 2\bar{\theta}_{200}\rho_1\rho_2 + \bar{\theta}_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\bar{\theta}_{020}\sigma_1\sigma_2 + \bar{\theta}_{300}\rho_1^3 \\ & + \bar{\theta}_{210}\rho_1^2\sigma_1 + \bar{\theta}_{120}\rho_1\sigma_1^2 + \bar{\theta}_{030}\sigma_1^3. \end{aligned}$$

In classifying the terms which belong to the expansion of R_3 and S_3 we bear in mind:

First. The θ_{ijk} in R_3 involve only cosines of even multiples of τ, except those which are coefficients of odd powers of σ (i. e., j is odd). These involve only sines of odd multiples. The reverse is the case in the $\bar{\theta}_{ijk}$ of S_3. If j is even, $\bar{\theta}_{ijk}$ involves only sines of odd multiples of τ. If j is odd, $\bar{\theta}_{ijk}$ involve only cosines of even multiples of τ.

Second. ρ_1 and ρ_2 are cosine series. The terms independent of λ involve only even multiples of τ.

σ_1 and σ_2 are sine series. The terms independent of λ involve only odd multiples of τ.

It is seen then that among the terms in R_3 which are independent of λ only cosines of even multiples of τ enter; and among the terms in S_3 which are independent of λ only sines of odd multiples enter. In the process of integration

therefore two types of non-periodic terms arise, first, those coming from the terms which involve the first multiple of $\lambda\tau$, and second, those coming from the terms independent of λ. It is important therefore to separate the various terms into three classes:

(a) terms independent of λ,

(b) terms involving first multiple of $\lambda\tau$ only,

(c) terms involving multiples of $\lambda\tau$ higher than the first.

The solution for these last terms is periodic. We rewrite then the differential equations (199) as follows:

$$\rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = \epsilon_3\theta_{001} + \epsilon_2 f_1(\lambda, \tau) + f_2(\lambda, \tau) + f_3(\tau) + f_4(\kappa\lambda, \tau),$$

$$\sigma_{33}'' + \theta_1\sigma_3 + \theta_3\rho_3 = \epsilon_2 g_1(\lambda, \tau) + g_2(\lambda, \tau) + g_3(\tau) + g_4(\kappa\lambda, \tau).$$

The coefficients of ϵ_2 are, explicitly,

$$(200)\qquad \begin{aligned} f_1(\lambda, \tau) &= \theta_{101}\alpha_2 + 2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\theta_{020}\gamma_1\gamma_6, \\ g_1(\lambda, \tau) &= 2\bar{\theta}_{200}\alpha_2\alpha_6 + \bar{\theta}_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\bar{\theta}_{020}\gamma_1\gamma_6. \end{aligned}$$

These terms are homogeneous in the first degree in α_2 and γ_1, and consequently involve only terms which carry the first multiple of $\lambda\tau$. They are of importance since they carry the undetermined constant ϵ_2 as a factor. The solution for these terms has, by Theorem II, the form

$$\rho = F_1(\lambda, \tau) + b_1\tau\alpha_1(\tau), \qquad \sigma = G_1(\lambda, \tau) + b_1\tau\gamma_2(\tau),$$

where $F_1(\lambda, \tau)$ and $G_1(\lambda, \tau)$ are periodic and involve only terms carrying the first multiple of $\lambda\tau$. b_1 is a constant depending upon $f_1(\lambda, \tau)$ and $g_1(\lambda, \tau)$. It is found, by a calculation not difficult, to have the value

$$b_1 = 3 + \mu^2\wp(\mu^2).$$

The functions $f_2(\lambda, \tau)$ and $g_2(\lambda, \tau)$ have the same characters as $f_1(\lambda, \tau)$ and $g_1(\lambda, \tau)$. They are considered separately since they are independent of ϵ_2. Their solutions may be written

$$\rho = F_2(\lambda, \tau) + b_2\tau\alpha_1(\tau), \qquad \sigma = G_2(\lambda, \tau) + b_2\tau\gamma_2(\tau).$$

The terms $f_3(\tau)$ and $g_3(\tau)$ are independent of λ. $f_3(\tau)$ contains only cosines of even multiples of τ, while $g_3(\tau)$ contains only sines of odd multiples of τ. The solution for these terms has the form

$$\rho = F_3(\tau) + b_3\tau\alpha_3(\tau), \qquad \sigma = G_3(\tau) + b_3\tau\gamma_4(\tau).$$

Finally, $f_4(k\lambda, \tau)$ and $g_4(k\lambda, \tau)$ involve only terms which carry multiples of $\lambda\tau$ higher than the first. The solution for these terms is

$$\rho = F_4(k\lambda, \tau), \qquad \sigma = G_4(k\lambda, \tau).$$

The entire solution is then

$$
\begin{aligned}
\rho_3 &= A^{(3)}\alpha_2(\tau) + D^{(3)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_3[\alpha_5(\tau) + a\tau\alpha_3(\tau)] \\
&\quad + \epsilon_2[F_1(\lambda, \tau) + b_1\tau\alpha_1(\tau)] + [F_2(\lambda, \tau) + b_2\tau\alpha_1(\tau)] \\
&\quad + [F_3(\tau) + b_3\tau\alpha_3(\tau)] + F_4(k\lambda, \tau), \\
\sigma_3 &= A^{(3)}\gamma_1(\tau) + D^{(3)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_3[\gamma_5(\tau) + a\tau\gamma_4(\tau)] \\
&\quad + \epsilon_2[G_1(\lambda, \tau) + b_1\tau\gamma_2(\tau)] + [G_2(\lambda, \tau) + b_2\tau\gamma_2(\tau)] \\
&\quad + [G_3(\tau) + b_3\tau\gamma_4(\tau)] + G_4(k\lambda, \tau).
\end{aligned} \tag{201}
$$

All of the functions $\alpha_i(\tau)$, $\gamma_i(\tau)$, $F_i(\tau)$, and $G_i(\tau)$ are periodic. In order that ρ_3 and σ_3 shall be periodic it is necessary and sufficient that the coefficient of $\tau\alpha_3(\tau)$ and $\tau\gamma_3(\tau)$, and the coefficient of $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$ be zero. That is,

$$D^{(3)} = -b_3 - a\epsilon_3,$$

and

$$\epsilon_2 = -\frac{b_2}{b_1},$$

by which condition the value of ϵ_2 is determined. In order to satisfy the initial conditions we must have $\rho_3 = 0$ at $\tau = 0$, and this determines $A^{(3)}$,

$$A^{(3)} = b_3\alpha_4(0) - \epsilon_3\alpha_6(0) + \frac{b_2}{b_1}F_1(0) - F_2(0) - F_3(0) - F_4(0).$$

Thus all the constants are determined except ϵ_3, and the solution is

$$
\begin{aligned}
\rho_3 &= A^{(3)}\alpha_2(\tau) - b_3\alpha_4(\tau) + \epsilon_3\alpha_6(\tau) - \frac{b_2}{b_1}F_1(\lambda, \tau) + F_2(\lambda, \tau) + F_3(\tau) + F_4(k\lambda, \tau), \\
\sigma_3 &= A^{(3)}\gamma_1(\tau) - b_3\gamma_3(\tau) + \epsilon_3\gamma_6(\tau) - \frac{b_2}{b_1}G_1(\lambda, \tau) + G_2(\lambda, \tau) + G_3(\tau) + G_4(k\lambda, \tau).
\end{aligned} \tag{202}
$$

The constant ϵ_3 will be determined in satisfying the periodicity condition for the coefficient of e^4. This process of integration can be continued indefinitely. ρ_3 and σ_3 have the same properties that have been stated for ρ_2 and σ_2. It is evident from the properties of the differential equations that these properties persist for ρ_4 and σ_4, and so on indefinitely. The coefficient for ϵ_{i-1} in so far as it carries the first multiples of $\lambda\tau$ is always the same as for ϵ_2. Therefore the constant ϵ_{i-1} can always be determined so as to avoid non-periodic terms of the type $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. The constant $D^{(i)}$ of integration can always be determined so as to destroy non-periodic terms of the type $\tau\alpha_3(\tau)$ and $\tau\gamma_4(\tau)$. The constant $A^{(i)}$ can always be determined so as to satisfy the initial conditions. The analysis of the type of terms entering is the same as for the subscript 3.

We have therefore a periodic solution with the period $2K\pi$ which is different from the class of generating orbits from which we set out, for the particle makes many revolutions before its orbit reënters. After integrating the equation $dv/d\tau = c/r^2$ the solution will contain five arbitrary constants (subject to the condition that λ must be rational), corresponding to mean distance, eccentricity, inclination, node, and epoch.

UNIVERSITY OF CHICAGO,
April 20, 1909.